

THE STRUCTURE OF TRANSLATION HALF RINGS IN EUCLIDEAN SPACES¹

PETER H. MASERICK²

1. Introduction. The concept of a half ring of sets was introduced by von Neumann [2], who made extensive use of it to define and investigate measure on product spaces. A half ring \mathfrak{R} is a collection of subsets of a given set satisfying the following conditions:

- (i) If $R, R' \in \mathfrak{R}$ then $R \cap R' \in \mathfrak{R}$.
- (ii) If $R, R' \in \mathfrak{R}$ and if $R \subset R'$ then there exists a finite chain $R_1, R_2, \dots, R_k, R_i \in \mathfrak{R}$ for $i=1, 2, \dots, k$ such that $R = R_1 \subset R_2 \subset \dots \subset R_k = R'$ and $R_j - R_{j-1} \in \mathfrak{R}$ for $j=2, 3, \dots, k$.

The family of all bounded intervals on the real line and the family of all parallelograms in the plane with parallel sides are examples of half rings.

We shall investigate properties of half rings in n -dimensional Euclidean space E^n ($n \geq 2$). In particular, for convex nondegenerate translation half rings as defined below, we shall give a complete characterization of all such families of sets in E^n . E^n is an n -dimensional Hilbert space and it will be assumed that the reader is familiar with the basic geometrical properties of such spaces, more specifically with such notions as hyperplanes, half spaces, spheres, line segments, cones, convex sets, support planes, translation of sets, etc.

A translation half ring (t.h.r.) in E^n is a half ring \mathfrak{R} of subsets of E^n such that if $R \in \mathfrak{R}$ and if R^* is a translate of R , then $R^* \in \mathfrak{R}$. If there exists a nondegenerate $R \in \mathfrak{R}$ (i.e. an element $R \in \mathfrak{R}$ whose interior R^0 is not empty), \mathfrak{R} is a nondegenerate t.h.r.

A convex polyhedron P is a subset of E^n satisfying:

- (a) P is bounded,
- (b) $P = \bigcap_{i=1}^k S_{H_i}$, where each S_{H_i} is an open or closed half space with the hyperplane H_i for its boundary.

If P is nondegenerate, the intersection $H_i \cap \bar{P}$ of H_i with the closure \bar{P} of P is defined to be a face of P provided there exists a sphere $S_i(x)$ contained in \bar{P} and tangent to H_i . The set of hyperplanes

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which determine faces of a nondegenerate polyhedron P also determines a minimal collection of half spaces whose intersection is P and which is contained in any other collection of half spaces which determine P . It also follows that $H_i \cap \bar{P}$ is a face of P if and only if there exists $x \in H_i \cap \bar{P}$ and a sphere $S_c(x)$ with center at x such that the disk $H_i \cap S_c(x)$ is a subset of $H_i \cap \bar{P}$. If P is degenerate, this can be used as the definition of a face of P . The faces $H_i \cap \bar{P}$ of P then generate unique hyperplanes.

If a family of hyperplanes consists of parallel hyperplanes, they are said to have the same direction d . A collection \mathcal{O} of convex polyhedra is oriented with respect to a set \mathcal{D} of directions if every $P \in \mathcal{O}$ has only faces which generate hyperplanes of directions $d \in \mathcal{D}$. The faces are said to have the direction of the respective hyperplanes they generate.

2. The structure of half rings in E^n .

THEOREM 1. *The set of all convex polyhedra forms a half ring in E^n .*

PROOF. The intersection of two convex polyhedra is again a convex polyhedron.

To prove the second half ring axiom let $P, P' \in \mathcal{O}$ and $P \subset P'$. P can be written as $\bigcap_{i=1}^{k-1} S_{H_i}$ where each $H_i \cap \bar{P} = F_i$ is a face of P . Let $P_1 = P, P_{i+1} = P_i \cup F_i$ for $i = 1, 2, \dots, k-1$. Clearly $P_k = \bar{P}$ and $\{P_i\}_{i=1,2,\dots,k}$ is a chain extension from P to \bar{P} which satisfies (ii). Thus without loss of generality we can assume $P = \bar{P}$. Therefore P can be expressed as $\bigcap_{i=1}^{k-1} \bar{S}_{H_i}$. Let $P_1 = P, P_j = (\bigcap_{i=j}^{k-1} \bar{S}_{H_i}) \cap P'$ for $j \leq k-1$ and $P_k = P'$. Each P_j is a convex polyhedron and $P_j \supset P_{j-1}$ for $j = 2, 3, \dots, k$. If $2 \leq j \leq k-1$ then

$$\begin{aligned} P_j - P_{j-1} &= \left(\bigcap_{i=j}^{k-1} \bar{S}_{H_i} \right) \cap P' - \left(\bigcap_{i=j-1}^{k-1} \bar{S}_{H_i} \right) \cap P' \\ &= \left(\bigcap_{i=j}^{k-1} \bar{S}_{H_i} - \bigcap_{i=j-1}^{k-1} \bar{S}_{H_i} \right) \cap P' = (E^n - \bar{S}_{H_{j-1}}) \cap \left(\bigcap_{i=j}^{k-1} \bar{S}_{H_i} \right) \cap P'. \end{aligned}$$

But $(E^n - \bar{S}_{H_{j-1}})$ is an open half space and so $P_j - P_{j-1} \in \mathcal{O}$. If $j = k$ similar reasoning will show that $P_k - P_{k-1} \in \mathcal{O}$. Therefore $\{P_i\}_{i=1,2,\dots,k}$ is the desired chain extension from $P_1 = P$ to $P_k = P'$.

COROLLARY. *If \mathcal{O} is the set of all convex polyhedra oriented with respect to a given set of directions then \mathcal{O} is a half ring.*

LEMMA. *If \mathcal{O} is a half ring of convex polyhedra, $P, P' \in \mathcal{O}$ such that $P \subset P', \{P_i\}_{i=1,2,\dots,k}$ is the finite chain of (ii) and if P_{i-1} is nondegenerate then $P_{i-1} = S_H \cap P_i$ where $H \cap \bar{P}_{i-1}$ is some face of P_{i-1} .*

PROOF. Since $P_{i-1} \subset P_i$ we can write $P_{i-1} = (\bigcap_{j=1}^l S_{H_j}) \cap P_i$. Assume that each $(\overline{P}_{i-1} \cap H_j) \cap P_i^0$ contains a point x_j such that $x_j \notin \bigcup_{t \neq j} H_t$. If $l > 1$ then the line segment $[x_1, x_2]$ contains an interior point z of P_{i-1} . For if $[x_1, x_2] \subset b(P_{i-1})$, since $[x_1, x_2] \subset P_i^0$ then for each $z \in [x_1, x_2]$ there must be an H_j for $j = 1, 2, \dots, l$ such that $z \in H_j$. But there are infinitely many points on $[x_1, x_2]$ and hence some H_j must contain at least two distinct points. But if a hyperplane contains two points of a line it contains the entire line. Thus we reach the contradiction that $x_1, x_2 \in H_j$ for some j .

Having shown the existence of $z \in [x_1, x_2]$ such that $z \in P_{i-1}^0$ we note that since $x_1, x_2 \in b(P_{i-1}) \cap P_i^0$ then $x_1, x_2 \in b(P_i - P_{i-1})$. But then by convexity we reach the contradiction that $z \in [x_1, x_2] \subset b(P_i - P_{i-1})$ or $z \in (P_i - P_{i-1}) \cap P_{i-1}^0$. Thus $l = 1$.

THEOREM 2. *If \mathcal{O} is a nondegenerate t.h.r. of convex polyhedra, then for every finite set $\{d_i\}_{i=1,2,\dots,m}$ of distinct directions of \mathcal{O} , \mathcal{O} contains a nondegenerate polyhedron R with exactly two faces of each such direction.*

PROOF. (a). It will be shown that for each direction d of \mathcal{O} , there exists a nondegenerate member of \mathcal{O} with direction d .

If P is degenerate and has direction d , since \mathcal{O} is nondegenerate there exists $P' \in \mathcal{O}$, $z \in E^n$ such that $z \in (P')^0$. Translate P into P^* such that $z \in P^*$. $P^* \cap P' = P_1$ can be extended into P' by a finite chain $\{P_i\}_{i=1,2,\dots,k}$ of (ii). If P_2 is nondegenerate and does not have direction d then $P_1 \cap P_2^0 \neq \emptyset$ (\emptyset is used to denote the null set). But since P_1 is degenerate, we reach the contradiction that $P_2 - P_1$ is not convex. If P_2 is degenerate then by repeatedly applying the above argument we can produce a nondegenerate P_i with direction d .

(b) Let P be a nondegenerate member of \mathcal{O} with a face $H \cap \overline{P}$ of direction d . Since \mathcal{O} is a t.h.r. we can assume the existence of a nondegenerate P' such that $(P')^0 \supset H \cap \overline{P}$. Let $\{P_i\}_{i=1,2,\dots,k}$ be the chain extension from P to P' . By the lemma there exists $i < k$ such that $P_{i-1} = S_H \cap P_i$ where $H \cap \overline{P}_{i-1}$ is a face of P_{i-1} . If we take the largest such i it is apparent that $P_i - P_{i-1}$ is nondegenerate. Thus P_{i-1} and $P_i - P_{i-1}$ are nondegenerate members of \mathcal{O} such that each of them have direction d and the intersection of their determining half spaces of these directions is not a half space no matter how these polyhedra are translated.

(c) Let $\{P_{ij}\}_{i=1,2,\dots,m}^{j=1,2}$ be a collection of polyhedra such that each P_{ij} has direction d_i with corresponding half spaces $S_{H_{ij}}$ where $S_{H_{i1}} \cap S_{H_{i2}}$ is not a half space. There exists $\epsilon > 0$ and $x_{ij} \in P_{ij}$ such that $S_\epsilon(x_{ij}) \subset P_{ij}$ and $S_\epsilon(x_{ij})$ is tangent to P_{ij} at H_{ij} . Since \mathcal{O} is a

t.h.r. we can assume that the x_{ij} 's coincide. From the definitions of a face of a polyhedron and direction stated in the introduction, it is clear that $\bigcap_{i=1; j=1,2}^m P_{ij}$ is the desired polyhedron.

If \mathcal{O} is a half ring of convex polyhedra oriented with respect to m distinct directions then since no convex polyhedra can have more than two faces of any given direction, it follows that no $P \in \mathcal{O}$ can have more than $2m$ faces. Examples can be constructed to show that not every half ring \mathcal{O} of convex polyhedra with an upper bound on the number of faces is oriented with respect to a finite number of directions. However, as the following corollary shows, this cannot occur in a nondegenerate t.h.r.

COROLLARY 1. *A nondegenerate t.h.r. \mathcal{O} of convex polyhedra is oriented with respect to a finite number of directions if and only if there exists a number m such that each $P \in \mathcal{O}$ has fewer than m faces.*

PROOF. The proof follows immediately from the foregoing statement and Theorem 2.

A nondegenerate polyhedron $R \subset E^n$ is a parallelepiped if R has exactly n distinct directions. It is known that a nondegenerate parallelepiped has exactly $2n$ faces. It is further known that every nondegenerate polyhedron has at least n distinct directions and hence a nondegenerate parallelepiped R , oriented with respect to these n directions exists. Moreover, R can be determined so that it will be contained within any arbitrarily small sphere. If a parallelepiped can be oriented with respect to n directions, these directions are said to be independent.

COROLLARY 2. *If \mathcal{O} is a nondegenerate t. h. r. in E^n and if $\{d_i\}_{i=1,2,\dots,n}$ is any set of n independent directions of \mathcal{O} , then there exists in \mathcal{O} a nondegenerate parallelepiped R with these n directions.*

PROOF. Let $\{P_{ij}\}_{i=1,2,\dots,n}^1,2$ be a collection of elements of \mathcal{O} constructed as in part (b) of the proof of Theorem 2 and described in part (c) of the same proof (P_{ij} has direction d_{ij}). Let R be a parallelepiped oriented with respect to $\{d_i\}_{i=1,2,\dots,n}$ such that for each P_{ij} , R can be translated into R_{ij} where R_{ij} will be contained in P_{ij} with its face of direction d_i in a corresponding face of P_{ij} . Translate each P_{ij} into P_{ij}^* so that the corresponding translates R_{ij}^* of R_{ij} will coincide. It is then clear that $\bigcap_{i=1; j=1,2}^n P_{ij}$ will be the desired parallelepiped.

COROLLARY 3. *The set \mathcal{R} of all parallelepipeds and degenerate polyhedra with respect to n directions is a half ring.*

PROOF. From Theorem 1 the set \mathcal{O} of all polyhedra oriented with

respect to these n directions is a half ring. But each nondegenerate $P \in \mathcal{O}$ must have at least n directions to be bounded. Therefore $\mathcal{O} = \mathcal{R}$.

COROLLARY 4. *\mathcal{O} is a nondegenerate t.h.r. of parallelepipeds and degenerate polyhedra oriented with respect to n distinct directions if and only if each nondegenerate $P \in \mathcal{O}$ has the same number of faces.*

PROOF. By Corollary 1, \mathcal{O} is oriented with respect to a finite number k of directions. By Theorem 2, there exists $P \in \mathcal{O}$ with at least $2k$ and hence exactly $2k$ faces. But by Corollary 2 there is $R \in \mathcal{O}$ with n directions and $2n$ faces. Therefore $2k = 2n$ or $n = k$ and hence \mathcal{O} is a set of parallelepipeds and degenerate polyhedra oriented with respect to n distinct directions.

3. The characterization of nondegenerate convex translation half rings.

THEOREM 3. *If \mathcal{G} is a t.h.r. of bounded convex subsets of E^n then the closure \bar{G} of each $G \in \mathcal{G}$ is a convex polyhedron.*

The proof of Theorem 3 will depend on the five lemmas included below. A boundary point x of a convex subset G is a *support point* if there is at most one support tangential hyperplane H containing x . It is known [1] that for each boundary point $y (y \in b(G))$ there is at least one support hyperplane H_y containing y .

LEMMA 1. *If the boundary point x of a convex set G is a boundary point for a sphere $S_\epsilon(y) \subset G$ then x is a support point for G .*

PROOF. It can be shown from Hilbert space considerations that every point of a sphere is a support point for the sphere. Therefore there is at most one hyperplane H_x supporting $S_\epsilon(y)$ at x . However there exists a hyperplane H'_x which is supporting for G at x and hence for $S_\epsilon(y)$ at x . Thus $H'_x = H_x$.

LEMMA 2. *If G is nondegenerate and convex and if $x \in b(G)$ then $\bar{S}_\epsilon(x)$ contains a support point of G for every $\epsilon > 0$.*

PROOF. The boundary $b(G)$ is closed and every closed sphere in E^n is compact. Since $x \in b(G)$ there exists $y \in S_{\epsilon/2}(x) \cap G^0$. Since $b(G)$ is closed, $b(G) \cap \bar{S}_\epsilon(x)$ is compact. Therefore the distance δ from y to $\bar{S}_\epsilon(x) \cap b(G)$ is assumed, say at $z \in \bar{S}_\epsilon(x) \cap b(G)$. Since the distance from x to y is less than $\epsilon/2$, $\delta < \epsilon/2$. But if $u \in b(G)$ and $u \notin \bar{S}_\epsilon(x)$ the distance from u to y is larger than $\epsilon/2$. Thus $S_\delta(y)$ is tangent to $b(G)$ at and only at points of $\bar{S}_\epsilon(x) \cap b(G)$ and hence $S_\delta(y) \subset G$. Therefore by Lemma 1, $S_\delta(x)$ must contain a support point of G .

LEMMA 3. *If G is closed, nondegenerate, convex and if $\{S_{H_x}\}$ is the collection of all support half spaces containing support points x then $\bigcap S_{H_x} = G$.*

PROOF. Clearly $\bigcap S_{H_x} \supset G$. If $z \in \bigcap S_{H_x}$ and $z \notin G$ then every line segment from z to an interior point of G contains a boundary point of G . Therefore the cone C determined by z and an interior sphere $S_\delta(v)$ of G contains a point $y \in b(G)$ on its axis. Hence there exists $\bar{S}_\epsilon(y) \subset C$. $\bar{S}_\epsilon(y) \cap b(G)$ contains no support point of G . For if w were such a point then since the support half space $S_{H_w} \supset G \supset S_\delta(v)$ and since there is a $u \in S_\delta(v)$ such that $w \in [z, u] \cap b(G)$ we have the contradiction that $z \notin S_{H_w}$. Hence $\bar{S}_\epsilon(y) \cap b(G)$ does not contain a support point which contradicts Lemma 2. So $z \in G$ and thus $\bigcap S_{H_x} = G$.

LEMMA 4. *If x and y are support points of a convex bounded set G such that x and y determine distinct support hyperplanes, then the line segment (x, y) (not including its endpoints) is contained in G^0 .*

PROOF. If $(x, y) \not\subset G^0$ then $z = x/2 + y/2 \in b(G)$. Thus z can be covered by a support hyperplane H_z . If H_z contains either x or y then it contains x and y . Therefore from hypothesis H_z contains neither x nor y . But $x \in b(G)$ and so $x \in S_{H_z}$. But then $y \notin S_{H_z}$ and this contradicts the fact that $y \in b(G)$.

LEMMA 5. *If G is a nondegenerate convex set but not a polyhedron then $b(G)$ contains an infinite sequence $\{x_n\}$ of distinct points such that the open segment $(x_i, x_j) \subset G^0$ for $i \neq j$.*

PROOF. If G is not a polyhedron then there exists an infinite number of support points (Lemma 3) each determining unique support hyperplanes. By Lemma 4 $(x_i, x_j) \subset G^0$ for $i \neq j$.

PROOF OF THEOREM 3. (a) If $G \in \mathfrak{G}$ is nondegenerate but not a convex polyhedron there exists an infinite sequence of points of $b(G)$ satisfying the conclusion of Lemma 5. Since \bar{G} is bounded, this sequence is bounded and therefore has a subsequence $\{x_i\}$ which converges to some point $x_0 \in b(\bar{G})$. Thus every sphere $S_\epsilon(x_0)$ with center at x_0 contains a convergent subsequence $\{x_{i_j}\}$ of $\{x_i\}$.

Translate G into G^* such that $x_0^* \in G^0$. The collection $\{G_t - G_{t-1}, G_1\}_{t=2,3,\dots,k}$, where each G_t is a member of the finite chain from $G_1 = G^* \cap G$ to G , is a partition of G . Each translate $x_{i_j}^*$ of x_{i_j} is such that $x_{i_j}^* \in b(G_t - G_{t-1})$ for some t . Since the sequence $\{x_{i_j}^*\}$ is infinite and $\{G_t - G_{t-1}\}$ is finite, there exists $m \geq 2$ and $z_1, z_2 \in \{x_{i_j}^*\}$ such that $z_1, z_2 \in b(G_m - G_{m-1})$. But since $(z_1, z_2) \subset G_1^0$, $(z_1, z_2) \not\subset (G_m - G_{m-1})$. But then (z_1, z_2) must be contained in the closure

of $G_m - G_{m-1}$ and hence we reach the contradiction that this closure contains an interior point of G_1 .

(b) If G is degenerate then unless G is either a line segment or a point, in which case G is a convex polyhedron, there exists $k < n$ such that G is a nondegenerate subset of E^k . Part (a) is then a valid argument showing that G is a polyhedron in E^k and hence G is a degenerate polyhedron in E^n .

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UNIVERSITY OF MARYLAND