AVERAGE INITIAL VELOCITY OF THE TERMINUS OF A TWO-DIMENSIONAL LINKAGE[1]

R. G. LANGEBARTEL

The problem to be treated concerns the motion in a plane of n particles connected successively by rigid rods freely hinged at the particles. The first rod is permanently attached to the origin about which it is free to pivot. Let the length of the ith rod be a_i and the angle between the radius vector to the ith particle (extended past the particle) and the $i+1$st rod, i.e. the difference between the angles of inclination of the ith radius vector and the $i+1$st rod, be θ_{i+1} with θ_1 the angle of inclination of the first rod. Let $r=a_1, r_2, r_3, \cdots, r_n$ and $\phi_1=\theta_1, \phi_2, \phi_3, \cdots, \phi_n$ be the polar coordinates of the n particles. Then the problem is to determine the mean initial velocity of the nth particle for any distribution of the initial pivotal angular velocities ω_i with the average taken over all configurations for which r_n initially has a given value r. This problem, suggested by Professor F. T. Wall, is of some interest in the dynamics of polymer chains such as in synthetic rubber theory. The problem of determining the probability that the terminus is initially at distance r from the origin can be regarded as a random walk problem and has been treated by Pearson [1],[2] Kluyver [2], Rayleigh [3], and Watson [4, p. 419]. The method to be used for handling the present kinematical problem is an extension of Kluyver's.

Let ρ_i be the initial value of r_i and let $\theta_i=\alpha_i+\omega_i t$. Denote by $P_n(r, a_i)$ the probability that $\rho_n<r$ and $E\{[\dot\phi_n(r, a_i)]_0\}$ the initial terminus angular velocity expectation for $\rho_n<r$. Thus,

$$E\{[\dot\phi_n(r, a_i)]_0\} = \frac{W_n(r, a_i)}{P_n(r, a_i)}$$

in which

$$W_n(r, a_i) = \frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}[\dot\phi_n]_0 F d\alpha_n\cdots d\alpha_1$$

where F is a discontinuous factor:

$$F = \begin{cases} 1, & \rho_n < r, \\ 0, & \rho_n > r. \end{cases}$$

Received by the editors December 9, 1957 and, in revised form, July 3, 1958.

[1] Prepared in part under a National Science Foundation Grant.

[2] Numbers in brackets refer to the references cited at the end of the paper.

The natural choice for F is $r\int_0^\infty J_1(r\tau)J_0(\rho_n\tau)d\tau$, [4, p. 406]. Differentiating with respect to time the relation

$$\tan(\phi_n-\phi_{n-1})=2a_nr_{n-1}\sin\theta_n/(r_n^2+r_{n-1}^2-a_n^2)$$

and noting $r_n^2=r_{n-1}^2+a_n^2+2a_nr_{n-1}\cos\theta_n$ gives the recursion formula so

$$\dot\phi_n = \dot\phi_{n-1} + a_n(r_{n-1}\dot\theta_n\cos\theta_n - \dot r_{n-1}\sin\theta_n + a_n\dot\theta_n)/r_n^2$$

so that $[\dot\phi_n]_0=[\dot\phi_{n-1}]_0+a_n(\rho_{n-1}\omega_n\cos\alpha_n-[\dot r_{n-1}]_0\sin\alpha_n+a_n\omega_n)/\rho_n^2$. The evaluation of W_n can then be carried out by successively interchanging the integrals and making use of this recursion formula for $[\dot\phi_n]_0$ and of the relations

$$\int_{-\pi}^{\pi}J_0(\rho_n\tau)d\alpha_n = 2\pi J_0(a_n\tau)J_0(\rho_{n-1}\tau),$$

$$\int_{-\pi}^{\pi}J_0(\rho_n\tau)\cos\alpha_n d\alpha_n = -2\pi J_1(a_n\tau)J_1(\rho_{n-1}\tau),$$

which follow from the Fourier series for $J_0(\rho_n\tau)$, [4, p. 358].

After one interchange of integrals the formula for W_n becomes

$$W_n(r,a_i) = \frac{r}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\int_0^\infty J_1(r\tau)\{[\dot\phi_{n-1}]_0I_1 + a_n\omega_nI_2 - a_n[\dot r_{n-1}]_0I_3\}d\tau d\alpha_{n-1}\cdots d\alpha_1,$$

$$\begin{cases} I_1 = \int_{-\pi}^{\pi} J_0(\rho_n\tau)d\alpha_n, \\ I_2 = \int_{-\pi}^{\pi}\dfrac{(a_n+\rho_{n-1}\cos\alpha_n)J_0(\rho_n\tau)}{\rho_n^2}d\alpha_n, \\ I_3 = \int_{-\pi}^{\pi}\dfrac{\sin\alpha_nJ_0(\rho_n\tau)}{\rho_n^2}d\alpha_n. \end{cases}$$

The value of I_1 is given above and I_3 vanishes by symmetry. The value of I_2 is perhaps most easily obtained by taking its derivatives with respect to τ:

$$\begin{aligned}(\tau I_2'(\tau))' &= -\tau\int_{-\pi}^{\pi}(a_n+\rho_{n-1}\cos\alpha_n)J_0(\rho_n\tau)d\alpha_n,\\ &= -2\pi\tau[a_nJ_0(a_n\tau)J_0(\rho_{n-1}\tau) - \rho_{n-1}J_1(a_n\tau)J_1(\rho_{n-1}\tau)],\\ &= -2\pi\frac{d}{d\tau}[\tau J_1(a_n\tau)J_0(\rho_{n-1}\tau)].\end{aligned}$$

Integrating this and taking into account the fact that all the integrals concerned are finite for $\tau=0$ gives the result

$$I_2(\tau) = -2\pi\int_0^\tau J_1(a_n\tau)J_0(\rho_{n-1}\tau)d\tau + C.$$

This constant of integration is found by setting $\tau=0$. The original integral for $I_2(0)$ can be evaluated by elementary methods and is

$$C = \begin{cases} 0, & a_n < \rho_{n-1}, \\ \dfrac{2\pi}{a_n}, & a_n > \rho_{n-1}, \\ \dfrac{\pi}{a_n}, & a_n = \rho_{n-1}, \end{cases}$$

$$= 2\pi\int_0^\infty J_1(a_n\tau)J_0(\rho_{n-1}\tau)d\tau.$$

Consequently,

$$I_2(\tau) = 2\pi\int_\tau^\infty J_1(a_n\sigma)J_0(\rho_{n-1}\sigma)d\sigma.$$

The change in the order of integration of the α_n and τ integrals is permitted by the uniform convergence of the infinite integral [4, p. 195; 5, pp. 23, 53]. There is an exceptional case, namely $\rho_{n-1}=a_n$ making $\rho_n=0$ for $\alpha_n=\pm\pi$. For the I_1 and I_2 terms the convergence ceases to be uniform in the neighborhood of these α_n values, but the integrals are boundedly convergent for these values [5, p. 43], thus again permitting the inversion. The singularity in the integrand of I_2 in this case is merely of the removable type since the fraction $(a_n+\rho_{n-1}\cos\alpha_n)/\rho_n^2$ reduces to $1/(2a_n)$ for $\rho_{n-1}=a_n$. However, the singularity in I_3 is more serious since, in fact, this integral does not converge if $\rho_{n-1}=a_n$. This result is evidently connected with the exceptional character of the angular coordinate at the origin in polar coordinates. This difficulty may be circumvented by summing the linkages symmetrically, thereby taking for I_3 the principal value, $\lim_{\epsilon\to 0}\int_{-\pi+\epsilon}^{\pi-\epsilon}(\quad)d\alpha_n$, which evidently is zero. Thus,

$$\int_{-\pi}^{\pi}[\phi_n]_0 J_0(\rho_n\tau)d\alpha_n$$

$$= [\phi_{n-1}]_0 2\pi J_0(a_n\tau)J_0(\rho_{n-1}\tau) + a_n\omega_n 2\pi\int_\tau^\infty J_1(a_n\sigma)J_0(\rho_{n-1}\sigma)d\sigma.$$

Proceeding in this way finally leads to the representation

$$W_n(r, a_i) = r \sum_{k=1}^{n} \omega_k a_k \int_0^\infty \int_\tau^\infty \prod_{q=1}^{k-1} J_0(a_q\sigma)J_1(a_k\sigma)d\sigma \prod_{p=k+1}^{n} J_0(a_p\tau)J_1(r\tau)d\tau.$$

In this formula the indicated products are to be suppressed if the upper limit is less than the lower. The infinite integral terms do not cause difficulty in the integral inversions since they are all of the same order for large τ as the remaining term, as can be seen by employing the asymptotic expansions for the Bessel functions and integrating by parts. Thus, for example, in the τ and α_{n-1} integral inversion the integrand is $O(\tau^{-3/2})$.

To complete the determination of $E\{[\dot{\phi}_n(r, a_i)]_0\}$ it remains but to use the formula of Kluyver [2] for $P_n(r, a_i)$, the probability that $\rho_n < r$:

$$P_n(r, a_i) = \frac{1}{(2\pi)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} F d\alpha_n \cdots d\alpha_1$$

$$= r \int_0^\infty \prod_{k=1}^{n} J_0(a_k\tau)J_1(r\tau)d\tau.$$

The resulting fraction for $E\{[\dot{\phi}_n(r, a_i)]_0\}$ simplifies considerably if all of the links are of the same length: $a_1 = a_2 = \cdots = a_n \equiv a$. In that case the σ integral in W_n can be integrated directly and W_n reduces to

$$r \sum_{k=1}^{n} \omega_k \frac{1}{k} \int_0^\infty J_0^n(a\tau)J_1(r\tau)d\tau = P_n(r, a) \sum_{k=1}^{n} \frac{1}{k} \omega_k.$$

Consequently, for equal links

$$E\{[\dot{\phi}_n(r, a)]_0\} = \sum_{k=1}^{n} \frac{1}{k} \omega_k.$$

Evidently, this same formula holds for the average for linkages (made up of equal links) whose termini initially are all at the same distance from the origin.

The integrands of $dW_n(r, a_i)/dr$ and $dP_n(r, a_i)/dr$ with the differentiation carried out under the integral sign are $O(\tau^{-(n-1)/2})$ as $\tau \to \infty$ where the constant implied in the order symbol is independent of r so long as r is bounded away from the origin. Accordingly, for such an r interval these integrals certainly converge uniformly for $n \geqq 4$ and the differentiation under the integral sign is valid [5, p. 59]. It follows that the initial angular velocity of the terminus averaged for linkages whose termini are initially at a distance r from the origin is

$$E\{[\dot{\Phi}_n(r, a_i)]_0\} = \frac{dW_n(r, a_i)/dr}{dP_n(r, a_i)/dr},$$

$$= \frac{\sum_{k=1}^{n} \omega_k a_k \int_0^\infty \int_\tau^\infty \prod_{q=1}^{k-1} J_0(a_q\sigma)J_1(a_k\sigma)d\sigma \prod_{p=k+1}^{n} J_0(a_p\tau)J_0(r\tau)\tau d\tau}{\int_0^\infty \prod_{k=1}^{n} J_0(a_k\tau)J_0(r\tau)\tau d\tau}$$

where $n \geqq 4$ and $r \geqq r_0 > 0$. As pointed out above this reduces to the simple formula given for $E\{[\dot{\phi}_n(r, a)]_0\}$ if $a_1 = \cdots = a_n \equiv a$, — an expression, it should be noted, that is independent of both r and a.

It is possible to use this same method to determine the average initial radial velocity of the terminus but it is not necessary to do so since symmetry considerations show that it is zero.

References

1. K. Pearson, *The problem of the random walk*, Nature, vol. 72 (1905) p. 294.

2. J. C. Kluyver, *A local probability problem*, Nederl. Akad. Wetensch. Proc. Ser. A. vol. 8 (1906) p. 341.

3. Lord Rayleigh, *On the problem of random vibrations and of random flights in one, two, or three dimensions*, Scientific Papers, vol. 6 (1920) p. 604.

4. G. N. Watson, *A treatise on the theory of Bessel functions*, 2d ed., Cambridge, 1945.

5. E. C. Titchmarsh, *The theory of functions*, 2d ed., Oxford, 1939.

University of Illinois