A DERIVATIVE FOR HAUSDORFF-ANALYTIC FUNCTIONS[1]

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1. **Introduction.** By a function on a hypercomplex system we shall mean a mapping whose domain and range are contained in the system. F. Hausdorff [2] proposed a definition for an analytic function on an (associative) hypercomplex system, over the complex field, with an identity, which may be stated as follows.

DEFINITION. The hypercomplex function $f=\sum_{k=1}^{n} f_k e_k$ (where the e_k, $k=1, \cdots, n$, are basis elements) is called an analytic function of $z=\sum_{k=1}^{n} z_k e_k$ at the point $z_0=\sum_{k=1}^{n} z_k^0 e_k$, if the (component) functions f_k, $k=1, \cdots, n$, are analytic functions of the complex variables $z_1, \cdots, z_n$ at the point $(z_1^0, \cdots, z_n^0)$ and if the differential

$$df = \sum_{k=1}^{n} df_k e_k = \sum_{j,k=1}^{n} \frac{\partial f_k}{\partial z_j} dz_j e_k,$$

where the $\partial f_k/\partial z_j$ are evaluated at $(z_1^0, \cdots, z_n^0)$, is a linear homogeneous function of the differential $dz=\sum_{k=1}^{n} dz_k e_k$; that is,

$$df = \sum_{j=1}^{r} u_j dz v_j,$$

where the u_j and v_j are hypercomplex variables that depend only on the point z_0.

The function f is said to be analytic in a domain of its variables if it is analytic at each point of the domain. Functions satisfying this definition will be called H-analytic.

The property of being H-analytic is independent of the basis, and it can be shown that if $f(z)$ and $g(z)$ are H-analytic, then $s(z)=f(z)+g(z)$ and $p(z)=f(z)\cdot g(z)$ are also H-analytic where $ds=df+dg$ and $dp=(df)\cdot g+f\cdot(dg)$ [6].

2. **Derivative.** We now extend the concept of H-analyticity to include a definition of a derivative as follows.

DEFINITION. If $f(z)$ is H-analytic in a domain D, then for all z contained in D, $df(z)=\sum_{k=1}^{r} u_k dz v_k$. We define the derivative of $f(z)$ with respect to z in D to be,

$$\frac{df(z)}{dz} = \sum_{k=1}^{r} u_k v_k.$$

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Although the u_k and v_k need not be unique (for example, setting $v_j = v_j' + v_j''$ will, in general, give a different set of u_k and v_k), if $\sum_{k=1}^{r} u_k dz v_k = df = \sum_{j=1}^{s} a_j dz b_j$, then by letting dz be the identity of the hypercomplex system we have $\sum_{k=1}^{r} u_k v_k = \sum_{j=1}^{s} a_j b_j$, that is,

THEOREM 2.1. *If $f(z)$ is H-analytic at the point z, then $df(z)/dz$ is unique.*

One can also show that this derivative has the properties stated above for the differential; namely, $ds/dz = df/dz + dg/dz$ and $dp/dz = (df/dz)\cdot g + f\cdot(dg/dz)$, where $s = f+g$, $p = f\cdot g$, and, f and g are H-analytic.

We now wish to apply this definition to functions on the algebra $\mathfrak{M}$ of square matrices of order n over the complex field.

THEOREM 2.2. *If $f(Z)$ is a function on the algebra $\mathfrak{M}$ whose component functions f_{ij}, $i, j = 1, \cdots, n$, are analytic functions, in some open domain, of the complex (component) variables z_{ij}, $i, j = 1, \cdots, n$ of Z, then*

(1) $f(Z)$ is H-analytic in a corresponding open domain of $\mathfrak{M}$, and

(2) $df(Z)/dZ = (\sum_{k=1}^{n} \partial/\partial z_{kk}) f(Z)$,

where $\partial f(Z)/\partial z_{kk}$ has the usual meaning of being the matrix whose i, j element is $\partial f_{ij}/\partial z_{kk}$.

PROOF. If we use as basis elements for the algebra $\mathfrak{M}$, the matrices E_{ij}, $i, j = 1, \cdots, n$, where E_{ij} is the $n\times n$ matrix which has a 1 in the i, j position and zeros elsewhere, then $df(Z) = \sum_{i,j,r,s=1}^{n} \partial f_{rs}/\partial z_{ij} dz_{ij} E_{rs}$ for $f(Z) = \sum_{r,s=1}^{n} f_{rs} E_{rs}$. For each p, $q = 1, \cdots, n$, let V_{pq} be the $n\times n$ matrix whose i, j element is given by $(V_{pq})_{ij} = \delta_{qi}\delta_{pj}$ and let U_{pq} be the $n\times n$ matrix whose i, j element is given by $(U_{pq})_{ij} = \partial f_{ip}/\partial z_{jq}$; then $df(Z) = \sum_{p,q=1}^{n} U_{pq} dZ V_{pq}$, where $dZ = \sum_{r,s=1}^{n} dz_{rs} E_{rs}$ that is, $f(Z)$ is H-analytic.[2] Also

$$\frac{df(Z)}{dZ} = \sum_{p,q=1}^{n} U_{pq} V_{pq}$$

$$= \sum_{i,j=1}^{n} \left(\sum_{k=1}^{n} \frac{\partial f_{ij}}{\partial z_{kk}} \right) E_{ij} = \left(\sum_{k=1}^{n} \frac{\partial}{\partial z_{kk}} \right) f(Z).$$

It can be seen from this that if two H-analytic matric functions have equal derivatives, the functions need not differ by a constant

[2] This result is also contained in a paper by Nicolò Spampinato, *Caratterizzazione delle funzioni di variabile ipercomplessa analitiche secondo Ringleb fra le funzioni a Derivata caratteristica*, Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 91–95. Edizioni Cremonense, Rome, 1942.

(matrix) but may differ by an arbitrary function whose component functions depend only on the off-diagonal variables z_{ij}, $i \neq j$.

Another definition for a derivative of a matric function, proposed by R. F. Rinehart [5] is as follows: A function $f(Z)$ on $\mathfrak{M}$, defined in a neighborhood $|z_{rs}-a_{rs}|<\delta$ of the matrix $A=(a_{rs})$, is called differentiable at $Z=A$, if, for all H in a sufficiently small neighborhood N of 0,

(i) the difference $f(A+H)-f(A)$ is expressible in the form $f(A+H)-f(A)=\sum_{i=1}^{k} P_iHQ_i$, where P_i and Q_i are in $\mathfrak{M}$,

(ii) $\lim_{H\to 0} \sum_{i=1}^{k} P_iQ_i$ exists.

If (i) and (ii) are fulfilled, then $\lim_{H\to 0} \sum_{i=1}^{k} P_iQ_i$ is called the derivative of $f(Z)$ at $Z=A$, and is denoted by $f^I(A)$. For later reference to Rinehart's paper we shall call matric functions that satisfy this definition, R-analytic. For this definition, analyticity of the component functions is not assumed, however the following theorem can be proved.

THEOREM 2.3. *If $f(Z)$ is H-analytic in a neighborhood of $Z=A$ in $\mathfrak{M}$, then $f(Z)$ is R-analytic at $Z=A$ and $f^I(A)=df(A)/dZ$, where $df(A)/dZ$ means $df(Z)/dZ$ evaluated at $Z=A$.*

PROOF. Since $f(Z)$ is H-analytic in a neighborhood N of $Z=A$,

$$df = \sum_{i,j,r,s} \frac{\partial f_{rs}}{\partial z_{ij}} dz_{ij}E_{rs} = \sum_{k=1}^{r} U_k dZ V_k, \tag{2.1}$$

where the U_k and V_k are independent of dZ and the $\partial f_{rs}/\partial z_{ij}$, $r, s, i, j =1, \cdots, n$, are evaluated at the components a_{ij} of A. Let $dZ=H =(h_{ij})$, then, for norm (H) sufficiently small such that $A+H$ is in N, (where, for convenience, the norm of any matrix $X=(x_{ij})$, $i=1, \cdots, m$, $j=1, \cdots, n$, with complex components x_{ij}, shall be defined by norm $(X)=\max_{i,j} |x_{ij}|$),

$$\Delta f = f(A + H) - f(A) = \sum_{k=1}^{r} U_kHV_k + \sum_{p,q=1}^{n} W_{pq}HV_{pq}, \tag{2.2}$$

where W_{pq} is the $n\times n$ matrix whose i, j element is given by $(W_{pq})_{ij} =\epsilon_{jq}^{ip}$, $\epsilon_{jq}^{ip}\to 0$ for each $p, q, i, j=1, \cdots, n$, as the $h_{km}\to 0$, $k, m, = 1, \cdots, n$, and the V_{pq} are as in Theorem 2.2 (since $\Delta f_{rs} = \sum_{i,j=1}^{n} \partial f_{rs}/\partial z_{ij}\, h_{ij}+ \sum_{i,j=1}^{n} \epsilon_{ij}^{rs}h_{ij}$, [3]). Thus, condition (i) of R-analyticity is satisfied. Since the U_i and V_i are independent of H, and since $\lim_{H\to 0} W_{pq}V_{pq}=0$, $f^I(A)$ exists, and

$$f^I(A) = \lim_{H\to 0}\left(\sum_{k=1}^{r} U_kV_k + \sum_{p,q=1}^{n} W_{pq}V_{pq}\right) = \sum_{k=1}^{r} U_kV_k = \frac{df(A)}{dZ}\,.$$

3. *H*-**analyticity of functions of a matrix arising from scalar functions of a complex variable.** In the established theory of matric functions arising from scalar functions there are several definitions for $f(Z)$, Z in $\mathfrak{M}$, where $f(z)$ is a scalar function of the complex variable z, all of which are essentially equivalent [4]. We will use the form of the definition proposed by Frobenius, which states that if a scalar function $f(z)$ is analytic at the eigenvalues $\lambda_1^0, \cdots, \lambda_n^0$ of $Z_0 = (z_{ij}^0)$ in $\mathfrak{M}$, then $f(Z_0)$ is defined by

$$f(Z_0) = \frac{1}{2\pi i}\int_C \frac{f(\lambda)}{\lambda I - Z_0}\, d\lambda, \tag{3.1}$$

where C is a set of admissible closed paths enclosing each at the distinct eigenvalues of Z_0. That is, the components of $f(Z_0)$ are the integrals, over C, of the corresponding components of the matrix $f(\lambda)(\lambda I - Z_0)^{-1}/2\pi i$.

We now wish to show that these functions defined by (3.1) are H-analytic and $df(Z_0)/dZ = f'(Z_0)$, where $f'(Z)$ is the matric function corresponding to $f'(z)$.

For all matrices Z sufficiently near Z_0 (that is, such that norm $(Z - Z_0) = \max|z_{ij} - z_{ij}^0|$ is sufficiently small), the eigenvalues $\lambda_1, \cdots, \lambda_n$ of Z will be near those of Z_0, since the zeros of a polynomial, in particular, det $(\lambda I - Z)$, are continuous functions of the coefficients [Weber's *Algebra*, vol. 1, §44] and the coefficients of det $(\lambda I - Z)$ are continuous functions (polynomials) of the elements z_{ij} of Z. Thus, we may choose a neighborhood N of Z_0, such that for Z in N, C also enclosed each of the distinct eigenvalues of Z. Then for all Z in N, $f(Z)$ is given by

$$f(Z) = \frac{1}{2\pi i}\int_C \frac{f(\lambda)}{\lambda I - Z}\, d\lambda.$$

The r, s element of the matrix $f(Z)$ is given by

$$f(Z)_{rs} = \frac{1}{2\pi i}\int_C f(\lambda) R_{rs}(\lambda, z_{ij}) d\lambda,$$

where $R_{rs}(\lambda, z_{ij})$ is the quotient of two polynomials in λ and the z_{ij}, $i, j = 1, \cdots, n$. Since C does not pass through any of the zeros of det $(\lambda I - Z)$, regardless of what Z in N is chosen, $f(\lambda)R_{rs}(\lambda, z_{ij})$ is a continuous function of the complex variables λ and z_{ij}, $i, j = 1, \cdots, n$, where each z_{ij} ranges over a region N_{ij} determined by N, and λ lies on C; also $R_{rs}(\lambda, z_{ij})$ is an analytic function of each z_{ij} in N_{ij} for every value of λ on C. Therefore $f(Z)_{rs}$ is an analytic function of each z_{ij}

of Z in N [Titchmarsh's *Theory of functions*, Chapter II, §2.83] and thus the $f(Z)_{rs}$ are analytic functions of the n^2 complex variables z_{ij} [1]. Hence, by Theorem 2.2, $f(Z)$ is H-analytic for Z in N.

Rinehart [5] has shown that if a scalar function $f(z)$ is analytic at the eigenvalues of a matrix Z_0, then $f(Z)$ is R-analytic at Z_0 and $f^I(Z_0)=f'(Z_0)$.

Hence, using Theorem 2.3, we have

THEOREM 3.1. *If $f(z)$ is a scalar function which is analytic at the eigenvalues of Z_0 in $\mathfrak{M}$, then the corresponding matric function $f(Z)$ is H-analytic in a neighborhood of Z_0 and $df(Z_0)/dZ=f'(Z_0)$.*

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