REMARKS ON COMPLEX CONTACT MANIFOLDS

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1. **Statement of results.** Let M be a complex manifold of complex dimension $2n+1$. Let $\{U_i\}$ be an open covering of M. We call M a *complex contact manifold* if the following conditions are satisfied:

(1) On each U_i there exists a holomorphic 1-form ω_i such that $\omega_i \wedge (d\omega_i)^n$ is different from zero at every point of U_i.

(2) If $U_i \cap U_j$ is nonempty, then there exists a nonvanishing holomorphic function f_{ij} on $U_i \cap U_j$ such that $\omega_i = f_{ij}\omega_j$ on $U_i \cap U_j$.

We shall prove the following

THEOREM. *If M is a complex contact manifold of complex dimension $2n+1$, then*

(1) *The structure group of the tangent bundle of M can be reduced to $U(1) \times (\mathrm{Sp}(n) \otimes U(1))$.*

(2) *Let $c_i(M)$ be the ith Chern class of (the tangent bundle of) M and α the characteristic class of the line bundle over M defined by $\{f_{ij}\}$. Then*

$$1 + c_1(M) + c_2(M) + \cdots = (1 + \alpha)(1 + n\alpha + \cdots).$$

In particular, $c_1(M)$ is divisible by $n+1$; $c_1(M) = (n+1)\alpha$.

(3) *There exists a principal fibre bundle P over M with structure group $U(1)$ such that P is a real contact manifold. Moreover, both P and a real contact form on P can be constructed in a natural way from M and ω_i.*

Chern has shown in [2] that the structure group of the tangent bundle of an orientable real contact manifold of real dimension $2n+1$ can be reduced to $SO(1) \times U(n) (= SO(1) \times (SU(n) \otimes U(1))$. Hence, (1) of our theorem is an analogue of the result of Chern.

At the end of this paper, we shall give two examples of (3).

2. **Proof of** (1).

Let $T_x(M)$ be the complex tangent space to M at a point x. Assume x to be in U_i. As $\omega_i \neq 0$ at x, $\omega_i = 0$ defines a $2n$-dimensional complex vector subspace F_x of $T_x(M)$. Let F be the vector bundle over M with fibres F_x. Let E be the line bundle $T(M)/F$. Then, $T(M) \cong F \oplus E$ (Whitney sum). From the definition of contact form, it follows that $d\omega_i$, on F_x, is of maximal rank and is defined up to a factor, i.e.,

$$d\omega_j \,|\, F_x = f_{ji} d\omega_i \,|\, F_x.$$

Received by editors May 31, 1958.

Now, (1) follows immediately from the definition of $\mathrm{Sp}(n)$. (See, for instance, [3].)

3. **Proof of** (2). From $\omega_i = f_{ij}\omega_j$, it follows easily that

$$\omega_i \wedge (d\omega_i)^n = (f_{ij})^{n+1}\omega_j \wedge (d\omega_j)^n.$$

Since $\omega_i \wedge (d\omega_i)^n$ is a holomorphic form of degree $2n+1$, $\{f_{ij}^{-(n+1)}\}$ defines the canonical bundle K of M. The characteristic class of K is $-c_1(M)$, (see for instance [1, §29]). The line bundle $E = T(M)/F$ is defined by $\{f_{ij}\}$. From $K = E^{-(n+1)}$, we obtain

$$c_1(M) = (n+1)\alpha$$

Let $c_i(F)$ be the ith Chern class of the vector bundle F. By the Whitney Duality Theorem,

$$1 + c_1(M) + c_2(M) + \cdots = (1 + \alpha)(1 + c_1(F) + c_2(F) + \cdots).$$

From $c_1(M) = (n+1)\alpha$, it follows that $c_1(F) = n\alpha$.

4. **Proof of** (3). Let L be the line bundle over M defined by $\{f_{ij}^{-1}\}$ and p the projection of L onto M. Let $h_i\colon p^{-1}(U_i) \to U_i \times C$ be the coordinate map. If $v \in L_x$ and $x \in U_i \cap U_j$, then

$$h_i(v) = (x, z_i), \qquad h_j(v) = (x, z_j), \qquad z_i = f_{ji}z_j.$$

Hence

$$z_i \cdot p^*(\omega_i) = z_j \cdot p^*(\omega_j),$$

showing that $\{z_i \cdot p^*(\omega_i)\}$ defines a holomorphic 1-form ω on L.

A simple calculation shows that

$$(\omega + \bar\omega) \wedge (d\omega + d\bar\omega)^{2n+1}$$
$$= A(z_i\bar z_i)^n(\bar z_i dz_i - z_i d\bar z_i) \wedge p^*(\omega_i \wedge (d\omega_i)^n \wedge \bar\omega_i \wedge (d\bar\omega_i)^n),$$

where $A = (2n+1)!/(n!)^2$.

We define a bundle P as follows. Let B_x be a positive definite hermitian form on L_x differentiable with respect to x. Let

$$P_x = \{v \in L_x;\ B_x(v, v) = 1\},$$

i.e., P_x is a circle in L_x. Then $P = \bigcup_{x \in M} P_x$ is a principal fibre bundle over M with group $U(1)$.

We shall show that the restriction of $\omega + \bar\omega$ to P defines a real contact structure on P. If P_x is given by

$$z_i\bar z_i = b(x)^2 \qquad (b(x) > 0),$$

then

$$(\bar z_i dz_i - z_i d\bar z_i) = 2\bar z_i dz_i - 2b \cdot db.$$

As $\omega_i \wedge (d\omega_i)^n \wedge \bar{\omega}_i \wedge (d\bar{\omega}_i)^n \wedge db$ is a form of degree $4n+3$ ($>$real dimension of M) on M, it vanishes identically. Hence, the restriction of $(\omega+\bar{\omega}) \wedge (d\omega+d\bar{\omega})^{2n+1}$ to P is

$$2Ab^{2n}\bar{z}_i dz_i \wedge p^*(\omega_i \wedge (d\omega_i)^n \wedge \bar{\omega}_i \wedge (d\bar{\omega}_i)^n),$$

which is clearly different from zero at every point of P.

REMARK. Let Q_x be the disk defined by the circle P_x. Then P is the boundary of $Q = \bigcup Q_x$ and $(\omega+\bar{\omega})$ is of maximal rank on Q except at the points of M, as easily verified. This agrees with a result of Gray [4].

5. Examples.

(1) *Complex projective spaces of odd dimension.* Let $z^1, z^2, \cdots, z^{2n+1}, z^{2n+2}$ be a coordinate in the $(2n+2)$-dimensional complex vector space C^{2n+2} and let $P_{2n+1}(C)$ be the $(2n+1)$-dimensional complex projective space. Then, $C^{2n+2} - \{0\}$ is the principal fibre bundle associated with a line bundle L over $P_{2n+1}(C)$. Set

$$\omega = z^1dz^2 - z^2dz^1 + \cdots + z^{2n+1}dz^{2n+2} - z^{2n+2}dz^{2n+1}.$$

Let $\{U_i\}$ be an open covering of $P_{2n+1}(C)$ and s_i a holomorphic cross-section of the principal bundle $C^{2n+2} - \{0\}$ over U_i. Set $\omega_i = s_i^*(\omega)$. Then $\{\omega_i\}$ defines a complex contact structure on $P_{2n+1}(C)$. Considering C^{2n+2} as the real $(4n+4)$-dimensional vector space R^{4n+4}, we obtain the $(4n+3)$-dimensional real projective space $P_{4n+3}(R)$. $P_{4n+3}(R)$ is a principal fibre bundle over $P_{2n+1}(C)$ with structure group $U(1)$. Every odd dimensional real projective space is a real contact manifold (see for instance [4]). The standard real contact form on $P_{4n+3}(R)$ is the one derived from the contact form of $P_{2n+1}(C)$ in the manner described in the proof of (3).

(2) *Complex projective co-tangent bundles*

Let V be a complex manifold of dimension $n+1$ and ω a holomorphic 1-form on the dual complex tangent bundle $\tilde{T}(V)$ (=the space of complex co-tangent vectors) defined by

$$\omega(u) = v(\delta\pi(u)), \qquad u \in T_v(\tilde{T}(V))$$

where π is the projection of $\tilde{T}(V)$ onto V and $\delta\pi$ is the differential of π; $\delta\pi$: $T(\tilde{T}(V)) \to T(V)$. In terms of local coordinate $z^0, z^1, \cdots, z^n$ of V and the induced coordinate $z^0, z^1, \cdots, z^n, \zeta_0, \zeta_1, \cdots, \zeta_n$ of $\tilde{T}(V)$,

$$\omega = \zeta_0 dz^0 + \zeta_1 dz^1 + \cdots + \zeta_n dz^n.$$

The fibre of $\tilde{T}(V)$ being the $(n+1)$-dimensional complex vector space, we construct in a natural way a fibre bundle over V whose fibre is the n-dimensional complex projective space (complex projective co-tangent bundle). This bundle of complex dimension $2n+1$ is our M. Considering the fibre of $T(V)$ as the $(2n+2)$-dimensional real vector space, we take as P the co-tangent sphere bundle over V (i.e., the fibre of P is a sphere in the fibre of $\tilde{T}(V)$).

$\tilde{T}(V)-V$ is the principal fibre bundle associated with a line bundle L over M. The definition of the complex contact structure on M is similar to the one in the first example. The classical real contact structure on the co-tangent sphere bundle P is the one derived from the complex contact structure on the complex projective co-tangent bundle M as described in the proof of (3).

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FIXED POINTS FOR MULTI-VALUED FUNCTIONS ON SNAKE-LIKE CONTINUA

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1. **Introduction.** A multi-valued function from a space X into a space Y is a point to set correspondence. Hamilton has shown that snake-like continua have the fixed point property with respect to maps [2]. Ward has extended this to show that snake-like continua have the fixed point property with respect to continuous multi-valued functions [6]. The results of this paper establish that a more general class of spaces, those which are inverse limits of arcs, have even stronger properties with respect to multi-valued functions.[2]

2. **Multi-valued functions.** All functions are multi-valued unless otherwise indicated. A map will always be a continuous single-valued function.

Presented to the Society, April 19, 1958; received by the editors June 5, 1958.

[1] National Science Foundation Fellow.

[2] The author wishes to thank Professor E. R. Fadell for his encouragement and for reading and criticizing this paper.