A REMARK ON CONTINUITY CONDITIONS[1]

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In a recent paper [1, p. 207], as a corollary to a theorem on Fourier series with gaps, I pointed out that a function of one real variable may satisfy a Lipschitz condition in a set of positive measure without satisfying such a condition in any interval. Here a Lipschitz condition in a general set of real numbers is defined as follows: $f(x) \in \text{Lip } \alpha$ in E if

$$f(x+h) - f(x) = O(|h|^{\alpha})$$

uniformly for x in E, as $h \to 0$ through *unrestricted* real values. When writing [1] I overlooked the fact that it is possible by simpler methods to obtain a much stronger result. Let E be a subset of $(0, 1)$, $f(x)$ a function defined in $(0, 1)$ and $\omega(t)$ a function defined, positive and monotonic increasing in $(0, 1)$ and satisfying $\omega(t) \to 0$ as $t \to 0$. Let us then call $\omega(t)$ a *modulus of continuity* of $f(x)$ in E if

$$|f(x+h) - f(x)| \leqq \omega(|h|) \tag{1}$$

for all x in E and all $x+h$ in $(0, 1)$ $(h \neq 0)$. With this terminology we have the following theorem.

THEOREM. *Let E be any subset whatever of* $(0, 1)$. *Let $\omega(t)$ be positive and monotonic increasing in* $(0, 1)$ *and satisfy $\omega(t) \to 0$ as $t \to 0$. Then there exists a function $f(x)$ defined in* $(0, 1)$ *such that*

(i) *$\omega(t)$ is a modulus of continuity of $f(x)$ in E,*

(ii) *$f(x)$ is discontinuous at every interior point of the complement of E.*

Here and in the rest of this note, "complement" means "complement with respect to $(0, 1)$." In the theorem we may, for instance, choose for E a nondense closed set of positive measure, whose complement is then an open set dense in $(0, 1)$. Thus we see from the theorem that $f(x)$ may satisfy as strong a continuity condition as we wish in E, without even being continuous in any interval.

To prove the theorem, denote by S the interior of the complement of E. We assume that S is nonempty, since if this is not the case the function $f(x) \equiv 0$ has all the required properties. Then S is the union of countably many disjoint open intervals, say

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$$S = \bigcup_{n=1}^{\infty} (x_n - \delta_n, x_n + \delta_n).$$

Let T denote the complement of S, so that $E \subset T$. Put $f(x) = 0$ in T, and also put $f(x) = 0$ when x is a rational point of S. If x is an irrational point of S, then $|x - x_n| < \delta_n$ for exactly one value of n; in this case put

$$f(x) = \omega(\delta_n - |x - x_n|).$$

Then $f(x)$ is defined throughout $(0, 1)$; we now prove (i) and (ii).

Suppose that $x \in E$ and $x + h \in (0, 1)(h \neq 0)$. If either $x + h \in T$ or $x + h$ is a rational point of S, then (1) is trivial. Otherwise $x + h$ is an irrational point of S, and so of some interval $(x_n - \delta_n, x_n + \delta_n)$, and

$$f(x + h) - f(x) = \omega(\delta_n - |x + h - x_n|). \tag{2}$$

But obviously $|h|$ is not less than the distance of $x + h$ from the nearer of the two points $x_n - \delta_n$, $x_n + \delta_n$; and this distance is $\delta_n - |x + h - x_n|$. Hence (1) is true, by (2) and the monotonicity of $\omega(t)$. This proves (i).

(ii) is plainly true, from the definitions of $\omega(t)$ and $f(x)$. This proves the theorem.

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Reference

1. P. B. Kennedy, *On the coefficients in certain Fourier series*, J. London Math. Soc. vol. 33 (1958) pp. 196–207.

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