ON THE SOLUTION OF ALGEBRAIC EQUATIONS[1]

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1. **Introduction.** A root, a, of an equation $f(x) = 0$ can be determined as a limit of a certain expression $Q_n(x)$ [2; 3; 4; 5], $a = \lim Q_n(x)$, as $n \to \infty$, where $Q_n(x)$ depends on $f(x)$ and its first n derivatives, and x is given an arbitrary value which is closer to a than to any other root of the equation. The purpose of this paper is to show some applications of the method to algebraic equations with real roots. A root can be evaluated with any accuracy by a simple procedure, and in many cases close upper and lower bounds of the root can be found at the same time. The main result of §2 is presented by formulas (15)–(18). Formula (22) of §3 represents a generalization of the Laguerre quadratic formula for the solution of algebraic equations with real roots. Formula (33) of §4 gives a generalization of the Laguerre quadratic formula for finding upper and lower bounds of the roots of an algebraic equation with real roots and, at the same time, of the Bernoulli method for finding the numerically largest root of such an equation.

2. **A few simple cases.** In this and in the following sections, let $f(x) = 0$ be an algebraic equation of degree N with real coefficients and real roots a_i, $i = 1, 2, \cdots, N$, $a_i \neq a_j$, for $i \neq j$. We may write the identity

(1)
$$1/f(x) = \sum A_i/(x - a_i)$$

and, by differentiating it $n - 1$ times, derive another identity

(2)
$$P_{n-1}/[f(x)]^n = \sum A_i/(x - a_i)^n,$$

where P_n is evaluated recursively [4; 5]:

(3)
$$P_n = f'P_{n-1} - f''fP_{n-2}/2! + \cdots + f^{(n)}(-f)^{n-1}P_0/n!$$

with $P_0 = 1$ and $f = f(x)$; and the sum is taken over all the roots of the equation.

Now if x is a number which is closer to a root a, which we shall write without the subscript, than to any other root of the equation, then

(4)
$$x - a = \lim fP_{n-1}/P_n,$$

and

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(5) $$x - a = \lim \left[f^n / P_n \right]^{1/n}, \quad \text{as } n \to \infty,$$

the nth root being appropriately chosen; it can be seen by the comparison with fP_{n-1}/P_n.

In the same way, if we take the identity

(6) $$f'(x)/f(x) = \sum (x - a_i)^{-1},$$

we derive

(7) $$D_n / [f(x)]^n = \sum (x - a_i)^{-n},$$

from which it follows $[4; 5]$ under the same conditions as in (4) and (5)

(8) $$x - a = \lim f D_{n-1} / D_n,$$

and

(9) $$x - a = \lim \left[f^n / D_n \right]^{1/n}, \quad \text{as } n \to \infty$$

with

(10) $$D_n = f' D_{n-1} - f'' D_{n-2}/2! + \cdots + f^{(n-1)}(-f)^{n-2} D_1/(n-1)!$$
$$+ f^{(n)}(-f)^{n-1} D_0/(n-1)!, \ n \geq 3,$$
$$D_0 = 1, \ D_1 = f', \ D_2 = f'^2 - f'' f.$$

By taking n large enough in (4), (5), (8) or (9) we can get an approximation to a with any required accuracy. We shall denote it by x_1 in each of the four cases and obtain

(11) $$x_1 = x - f P_{n-1}/P_n,$$

(12) $$x_1 = x - (f^n/P_n)^{1/n},$$

(13) $$x_1 = x - f D_{n-1}/D_n,$$

(14) $$x_1 = x - (f^n/D_n)^{1/n}.$$

The following statements give an account of the character of the approximations x_1 to a root of $f(x) = 0$ in some particular (see also $[6]$) cases.

1°. *If x is an upper bound of the roots of the equation $f(x) = 0$, and a_N is its largest root, then*

(15) $$a_N < x - f P_{n-1}/P_n,$$

(16) $$a_N < x - f/P_n^{1/n},$$

(17) $$a_N > x - f D_{n-1}/D_n, \qquad n \geq 2$$

(18) $$a_N < x - f/D_n^{1/n}.$$

$2°$. *If n is even, then the interval $(x-f/D_n^{1/n},\ x+f/D_n^{1/n})$ is free of roots of the equation; and the interval $(x-fN^{1/n}/D_n^{1/n},\ x+fN^{1/n}/D_n^{1/n})$ has in it at least one root of the equation, and only one if n is large enough and x is not the mid-point of the two adjacent roots.*

In case of the Equations (11), (12), and (14) we can write by using (2) and (7), $x-a_N=(x-x_1)(1+\alpha_n)$, where α_n is a positive non-increasing function of n. From this fact inequalities (15), (16), and (18) follow. In case of (13) we may write

$$x - a_N = (x - x_1)(1 + \beta_n)/(1 + \beta_{n-1}),$$

where β_n is positive, decreasing with n. Therefore, $a_N > x_1$, and this proves (17). Statement $2°$ follows immediately by a similar argument.

3. **A generalization of the Laguerre quadratic formula.** Let n be an even number and u an arbitrary real number not equal to x or a root of the equation. We may write the identity

$$(19) \qquad \begin{aligned} &(u - x)^n f'(x)/f(x) \\ &\qquad = \sum (u - a_i)^n/(x - a_i) + Nx^{n-1} + c_1 x^{n-2} + \cdots + c_n \end{aligned}$$

and by differentiating it $n-1$ times derive

$$(20) \qquad Q_n(x, u)/[f(x)]^n = \sum (u - a_i)^n/(x - a_i)^n,$$

where the sum is taken over all the roots of the equation $f(x)=0$, and $Q_n(x, u)$ is evaluated by the formula

$$(21) \qquad Q_n(x, u) = \sum_{k=0}^{n-1} \binom{n}{k} (u - x)^{n-k} D_{n-k}f(x)^k + N.$$

We can now prove that

$3°$. *If x is an arbitrary real number, then the two real roots, X_1 and X_2, of the equation*

$$(22) \qquad Q_n(x, u)/[f(x)]^n = (u - X)^n/(x - X)^n$$

are separated by x and the interval (X_1, X_2) has no roots of $f(x)=0$ in it. Further, at least one of these roots converges to a root of the equation $f(x)=0$, as $n\to\infty$. With a finite n the accuracy of the approximation X to a root a increases with the increase of n, and it depends also on the choice of u.

For a proof we shall consider the equation

$$(23) \qquad \sum (u - a_i)^n/(x - a_i)^n - (u - X)^n/(x - X)^n = 0,$$

which is identical with (22). The left member of this equation is posi-

tive for X equal to any root of $f(x) = 0$, because the last term in it is equal to only one term of the sum, all of whose terms are positive, but if X is close enough to x, then the last member becomes larger than the total sum and the left side of (23) will be negative. This proves the first half of the statement. If u satisfies the equation $(u-a_1)/(x-a_1) = -(u-a_2)/(x-a_2)$, where a_1 and a_2 are the two roots adjacent to x, $a_1 < X_1 < x < X_2 < a_2$, then the sum in (23) has two dominant terms, $(u-a_1)^n/(x-a_1)^n$ and $(u-a_2)^n/(x-a_2)^n$. Therefore X_1 and X_2 will approach a_1 and a_2 with the increase of n. If $x-a_1 > a_2-x$, then the two dominant terms are obtained by taking $u = [(a_1+a_2)x - 2a_1a_2]/(2x-a_1-a_2)$. Any u in the interval $x < u < [(a_1+a_2)x-2a_1a_2]/(2x-a_1-a_2)$ makes the dominant term the one involving a_1; and the two intervals, $-\infty = u < x$ and

$$[(a_1 + a_2)x - 2a_1a_2]/(2x - a_1 - a_2) < u = \infty,$$

define u making the dominant term the one involving a_2. If $x-a_1 = a_2-x$, then $u > x$ makes the dominant term the one involving a_1, and $u < x$ makes the dominant term the one involving a_2. All possible situations are illustrated by diagrams 1 and 2.

Two values u_1 and u_2 necessary for approximating a_1 and a_2 can be selected by first applying formula (14) for determining an interval (x_1', x_1'') around x free of roots. Afterwards, u_1 can be taken to be any number in the interval $x < u \leq 2x_1'' - x$, and u_2 any number in the interval $2x_1' - x \leq u < x$. Both u_1 and u_2 would fall in the intervals defined above; however, they would not necessarily insure the best approximations possible by (22) with a fixed n.

Denoting now

(24) $$S_n = Q_n(x, u)/[f(x)]^n,$$

we get from (22)

(25) $$X = x - (x - u)/(1 \pm (S_n)^{1/n})$$

the nth root being taken with plus for X_1 when $u < x$, and with minus when $u > x$. A particular case, $u \to \infty$, reduces formula (25) to (14).

In the particular case $n = 2$, the best approximations to the roots a_1 and a_2 can easily be obtained as a simple problem in extrema. Two equations determining X_1, X_2 and u_1, u_2 are as follows:

(26) $$(N - 1)(L^2D_2 - f^2) - (LD_1 - f)^2 = 0,$$

or

(26$_1$) $$[(N - 2)f'^2 - (N - 1)ff'']L^2 + 2ff'L - Nf^2 = 0,$$
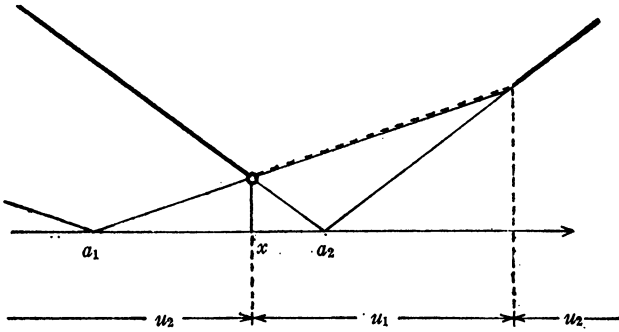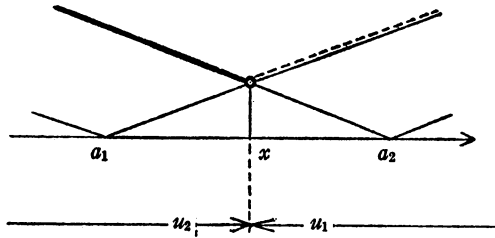
DIAGRAM 1

DIAGRAM 2

and

(27) $$f''m^2 + 2(N - 1)f'm + N(N - 1)f = 0,$$

where $L = x - X$, and $m = u - x$.

From these equations

(28) $$L = \frac{Nf}{f' \pm H^{1/2}},$$

and

(29) $$m = -\frac{N(N - 1)f}{(N - 1)f' \mp H^{1/2}},$$

where $H = (N-1)^2 f'^2 - N(N-1)ff''$.

A few relations between L and m are helpful in calculations, namely:

(30) $$L = mf/[mf' + (N - 1)f],$$

(31) $$m = -L(N - 1)f/[Lf' - f],$$

and

(32) $$1/L + 1/m = \pm H^{1/2}/(N-1)f.$$

Equation (26) is the famous quadratic formula suggested by Laguerre [7, p. 97–103] and derived by Hermite [7, p. 461].

We do not know, however, a simple equation defining u which would give the best approximations in the general case (25). Before applying the general formula (22), it is useful to know the results of applying (26) and (27) in deciding what value of u should be used. The u and x values could be selected such as to make the calculations as simple as possible. This makes the use of formula (25) preferable to the repeated applications of the Laguerre quadratic formula.

4. A generalization of the Bernoulli method. While the value of x in the preceding section could also be taken as an upper or lower bound of the roots of the equation $f(x)=0$, a particular case $x\to\infty$ deserves a special treatment.

If in (22) and (23) we put $x\to\infty$, we get

(33) $$Q_n(u) = (u-X)^n,$$

and

(34) $$\sum (u-a_i)^n - (u-X)^n = 0,$$

respectively, where

(35) $$Q_n(u) = Nu^n + \sum_{k=1}^{n}\binom{n}{k} D_k'.$$

Here

(36) $$D_k' = \begin{vmatrix} p_1 & 1 & 0 & \cdots & - & - & \cdots & - \\ 2p_2 & p_1 & 1 & \cdots & - & - & \cdots & - \\ - & - & - & \cdots & - & - & \cdots & - \\ Np_N & p_{N-1} & p_{N-2} & \cdots & - & - & \cdots & - \\ 0 & p_N & p_{N-1} & \cdots & - & - & \cdots & - \\ - & - & - & \cdots & - & - & \cdots & - \\ 0 & 0 & 0 & \cdots & p_N & p_{N-1} & \cdots & p_1 \end{vmatrix}$$

if $f(x) = x^N + p_1 x^{N-1} + p_2 x^{N-2} + \cdots + p_N$. It can be evaluated recursively

(37) $$D_k' = p_1 D_{k-1}' - p_2 D_{k-2}' + \cdots + (-1)^{k-2} p_{k-1} D_1' + (-1)^{k-1} k p_k D_0'$$
$$D_0' = 1, \quad D_1' = p_1, \quad D_2' = p_1^2 - 2p_2;$$

or by an explicit expression

$$(38) \quad D_k' = k \sum (-1)^{i+j+\cdots+r-1} \frac{(i+j+\cdots+r-1)!}{i!j!\cdots r!} p_q^i p_s^j \cdots p_t^r,$$

the sum being taken over all the terms for which $qi+sj+\cdots+tr = k$.

Now, if a_1 and a_N denote the smallest and the largest roots of the equation $f(x)=0$, remembering that n is even, the following conclusions can be made as to the roots of (33).

4°. *If u is an arbitrary real number, then the two real roots X_1 and X_2, $X_1 < X_2$, of the Equation (33) are separated by all the roots of the equation $f(x)=0$. If $u = (a_1+a_N)/2$, then X_1 converges to a_1, and X_2 to a_N, as $n \to \infty$; if $u > (a_1+a_N)/2$, then X_1 converges to a_1, and if $u < (a_1+a_N)/2$ then X_2 converges to a_2 as $n \to \infty$.*

The first half of the statement follows from the fact that the last term in (34) equals or is less than one term of the sum so long as $a_1 \leqq X \leqq a_N$, but it is larger than the sum for some $X < a_1$ and $X > a_N$. Further, if $u - a_1 = a_N - u$, there are two dominant terms in the sum, and therefore X_1 and X_2 approach a_1 and a_N respectively as n increases; if $u - a_1 > a_N - u$, then only one term is dominant, $(u-a_1)^n$, and therefore X_1 converges to a_1; for the same reason if $u - a_1 < a_N - u$, X_2 converges to a_N. The particular case $u = 0$ reduces (33) to

$$(39) \qquad\qquad D_n' = X^n,$$

the result which was suggested by Bernoulli [1] for approximating the root which is largest in absolute value.

The values of u_1 and u_2 giving the best approximations to a_1 and a_N can easily be found for $n = 2$ as a problem in extrema. The equation defining these values is

$$(40) \qquad N(N-1)u^2 + 2(N-1)p_1 u + 2p_2 = 0,$$

and the Equation (33) becomes

$$(41) \qquad NX^2 + 2p_1 X + 2(N-1)p_2 - (N-2)p_1^2 = 0.$$

From these equations

$$(42) \qquad u = -[(N-1)p_1 \pm (H_1)^{1/2}]/N(N-1)$$
$$= -2p_2/[(N-1)p_1 \mp (H_1)^{1/2}],$$

$$(43) \qquad X = (-p_1 \pm (H_1)^{1/2})/N$$
$$= [-(N-2)p_1^2 + 2(N-1)p_2]/(-p_1 \mp (H_1)^{1/2}),$$

where $H_1 = (N-1)^2 p_1^2 - 2N(N-1)p_2$. The upper sign before the radical corresponds to the largest root. The following two relations between u and X are useful in applications:

$$(44) \qquad (N-1)u + X + p_1 = 0,$$

and

$$(45) \qquad u - X = \pm (H_1)^{1/2}/(N-1).$$

The values of u determined by (40), and rounded off for convenience, can be used for the general formula (33). Formula (41) represents another Laguerre quadratic equation [7, p. 101] for evaluating upper and lower bounds of the roots of algebraic equations with real roots.

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