NOTE ON A THEOREM OF FUGLEDE AND PUTNAM

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1. An involution in a ring \( A \) is a mapping \( a \rightarrow a^* \) \((a \in A)\) such that \( a^{**} = a \), \((a + b)^* = a^* + b^*\), \((ab)^* = b^*a^*\). An element \( a \in A \) is (1) normal if \( a^*a = aa^* \), (2) self-adjoint if \( a^* = a \), (3) unitary if \( a^*a = a^*a = 1 \) \((1 = \text{unity element of } A)\). We say that "Fuglede's theorem holds in \( A \)" in case the relations \( a \in A \), \( a \) normal, \( b \in A \), \( ba = ab \), imply \( ba^* = a^*b \); briefly, \( A \) is an FT-ring.

It follows from a theorem of B. Fuglede that the ring \( A \) of all bounded operators in a Hilbert space (hence any adjoint-containing subring thereof) is an FT-ring [3, Theorem I]. For this ring, C. R. Putnam obtained the following generalization [9, Lemma]: if \( a_1, a_2 \) are normal, and \( ba_1 = a_2b \), then \( ba_1^* = a_2^*b \). A ring with involution, in which the latter theorem holds, will be called a PT-ring.

We denote by \( A_n \) the ring of all \( n \times n \) matrices \( x = (a_{ij}), a_{ij} \in A \), provided with the "conjugate-transpose" involution \( x^* = (a_{ji}) \).

**Theorem 1.** If \( A_2 \) is an FT-ring, then \( A \) is a PT-ring.

**Proof.** Suppose \( a_1, a_2 \) are normal elements of \( A \), and \( ba_1 = a_2b \). Define
\[
x = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}.
\]
Clearly \( x \) is normal. Moreover,
\[
yx = \begin{pmatrix} 0 & 0 \\ ba_1 & 0 \end{pmatrix}, \quad xy = \begin{pmatrix} 0 & 0 \\ a_2b & 0 \end{pmatrix}
\]
thus \( yx = xy \). Since Fuglede's theorem holds in \( A_2 \), \( yx^* = x^*y \), in other words \( ba_1^* = a_2^*b \).

**Example 1.** Let \( A \) be an involutive (i.e. adjoint-containing) ring of bounded operators acting on a Hilbert space \( H \). Then \( A_2 \) is an involutive ring acting on the direct sum of two copies of \( H \). By Fuglede's theorem, \( A_2 \) is an FT-ring; thus \( A \) is a PT-ring by Theorem 1. This is Putnam's generalization of the Fuglede theorem [9, Lemma]. The argument extends easily to cover the case that \( a_1, a_2 \) are possibly unbounded. The result then reads: if \( ba_1 \subseteq a_2b \) then \( ba_1^* \subseteq a_2^*b \).

In the reverse direction, if \( A \) is a PT-ring, then Fuglede's theorem

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holds for the diagonal normal elements of $A_2$; we omit the obvious proof:

**Theorem 2.** If $A$ is a PT-ring, and $a_1, a_2 \in A$ are normal, then the commutant of the normal matrix

$$x = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

in $A_2$ is involutive; that is, the relations $y \in A_2, yx = xy$, imply $yx^* = x^*y$.

A ring $A$ with involution is said to satisfy the *square root axiom* [6, Chapter VII] in case: given any $a \in A$, there exists a self-adjoint element $r$ such that $r^2 = a^*a$, and such that $r$ is in the double commutant of $a^*a$ (that is, the relation $b(a^*a) = (a^*a)b$ implies $br = rb$).

**Examples:** any $C^*$-algebra (see [7, Theorem 26A]); the regular ring of a finite AW*-algebra [1, Corollary 6.2]. Suppose $A$ is a ring satisfying the SR-axiom, and $a \in A$ is invertible. Write $u = ar^{-1}$, where $r$ is the self-adjoint described above; clearly $u^*u = uu^* = 1$. The factorization $a = ur$ is called a “polar decomposition” for $a$.

**Theorem 3.** Let $A$ be a PT-ring satisfying the square-root axiom. If $a_1, a_2$ are similar normal elements, they are unitarily equivalent.

**Proof.** Suppose $ba_1b^{-1} = a_2$. Then $ba_1 = a_2b$; since $A$ is a PT-ring, $ba_1^* = a_2^*b$, thus $a_1b^* = b^*a_2$. Let $b = ur$ be a polar decomposition. Then $a_1$ commutes with $b^*b$; for, $a_1(b^*b) = (a_1b^*)b = (b^*a_2)b = b^*(a_2b) = b^*(ba_1) = (b^*b)a_1$. Hence $a_1 = ra_1$, and $a_2 = ba_1b^{-1} = (ur)a_1(r^{-1}u^*) = ua_1r^{-1}u^* = ua_1u^*$.

**Example 2 (Putnam).** If $A$ is the ring of all bounded operators in a Hilbert space, and $a_1, a_2 \in A$ are similar normal operators, then $a_1, a_2$ are unitarily equivalent by Example 1 and Theorem 3 (see [9, Theorem 1]). The argument works just as well for $A$ any $C^*$-algebra, the point being that the elements implementing the similarity and unitary equivalence are to be drawn from $A$.

A ring $A$ with involution is said to possess a *trace* if there exists a mapping $a \mapsto \text{tr}(a)$ of $A$ into some abelian group, such that (1) $\text{tr}(a + b) = \text{tr}(a) + \text{tr}(b)$, (2) $\text{tr}(ab) = \text{tr}(ba)$, and (3) $\sum_1^k \text{tr}(a_i^*a_i) = 0$ implies $a_1 = \cdots = a_k = 0$.

**Theorem 4.** If $A$ is a ring with involution and trace, then $A$ is a PT-ring.

**Proof.** Since $A_2$ also has a trace, defined for a matrix $x = (a_{ii})$ by the formula $\text{tr}(x) = \sum_1^n \text{tr}(a_{ii})$, it will suffice by Theorem 1 to show that $A$ is an FT-ring. Suppose $x$ is normal, and $yx = xy$. It must be
shown that \( z = yx^* - x^*y \) is 0. We learned the ensuing argument for this from I. Kaplansky. One has
\[
zz^* = yx^*xy^* - yx^*y^*x - x^*yy^*x + x^*yy^*x
= yxx^*y^* - yx^*y^*x - x^*yy^*x + x^*yy^*x
= xyx^*y^* - yx^*y^*x - xx^*yy^* + x^*yy^*x.
\]
Since \( \text{tr}(xyx^*y^*) = \text{tr}(yx^*y^*x) \), and \( \text{tr}(xx^*yy^*) = \text{tr}(x^*yy^*x) \), one has 
\( \text{tr}(zz^*) = 0 \), hence \( z = 0 \).

Example 3. Let \( A \) be a commutative ring with involution, such that \( \sum_i a_i^*a_i = 0 \) implies \( a_1 = \cdots = a_k = 0 \), and set \( \text{tr}(a) = a \). Then \( A_n \) is a PT-ring by Theorem 4.

Example 4. Let \( Q \) be the ring of all real quaternions \( a = \alpha + \beta i + \gamma j + \delta k \), with involution \( a^* = \alpha - \beta i - \gamma j - \delta k \). One has \( a^*a = aa^* = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \), so that incidentally every element of \( Q \) is normal. Set \( \text{tr}(a) = \alpha \). It results from Theorem 4 that \( Q_n \) is a PT-ring. This is Putnam's theorem for finite-dimensional quaternionic Hilbert space, and raises the analogous question for infinite dimension.

Example 5. Let \( A \) be a homogeneous AW*-algebra of finite order \( n \), so that \( A = Z_n \), where \( Z \) is the center of \( A \). Let \( C \) be the regular ring of \( A \), \( W \) the regular ring of \( Z \); we may identify \( W \) with the center of \( C \) [1, Theorem 9.2]. Now, \( W \) has the properties in Example 3 [1, Lemma 3.4]; since \( C = W_n \) [2, concluding remark (2)], it follows that \( C \) has a \( W \)-valued trace. Thus \( C \) is a PT-ring. See Theorem 5 for the generalization to \( A \) of finite Type I.

Lemma. Suppose \( A \) is the C*-sum of a family \( (A_i) \) of finite AW*-algebras, \( C \) is the regular ring of \( A \), and \( C_i \) is the regular ring of \( A_i \). Then \( C \) is the complete direct sum of the \( C_i \).

Proof. According to [5, §2], \( A \) is the set of all families \( a = (a_i) \) with \( a_i \in A \) and \( ||a_i|| \) bounded; the operations in \( A \) are coordinate-wise. One knows from [5] that \( A \) is an AW*-algebra, and is clearly of finite class, so that we may speak of its regular ring \( C \).

Let \( D \) be the complete direct sum of the \( C_i \). That is, \( D \) is the set of all families \( x = (x_i) \) with \( x_i \in C_i \), with the coordinatewise operations. By an easy coordinatewise argument, one sees that \( D \) is regular. It must be shown that \( D = C \).

We may identify \( A \) as an involutive subalgebra of \( D \). We shall prove \( D = C \) by verifying the criterion of [1, §11]. Suppose \( x, y, z \in D \), and \( x^*x + y^*y + z^*z = 1 \). Then \( x_i^*x_i + y_i^*y_i + z_i^*z_i = 1 \) for all \( i \), hence \( x_i, y_i, z_i \in A_i \); since these elements all have norm \( \leq 1 \), one has \( x, y, z \in A \).
Theorem 5. If $A$ is a finite $AW^*$-algebra of Type I, its regular ring $C$ possesses a center-valued trace. In particular, $C$ is a $PT$-ring.

Proof. Write $A$ as the $C^*$-sum of a family $(A_i)$ of homogeneous algebras, and let $C_i$ be the regular ring of $A_i$. By the Lemma, $C$ is the complete direct sum of the $C_i$. It follows at once that the center $W$ of $C$ is the complete direct sum of the centers $W_i$ of $C_i$. According to Example 5, $C_i$ has a $W_i$-valued trace. Then $(x_i) \mapsto (\text{tr } x_i)$ defines a $W$-valued trace on $C$, thus $C$ is a $PT$-ring by Theorem 4.

It is reasonable to suppose that $C$ is a $PT$-ring, for any finite $AW^*$-algebra $A$; in any case, since $A_2$ is $AW^*$ with regular ring $C_2$ by [2], it would suffice by Theorem 1 to show that $C$ is an $FT$-ring.

Corollary. Let $A$, $C$ as in Theorem 5. If $z_1$, $z_2$ are similar normal elements of $C$, they are unitarily equivalent.

Proof. $C$ is a $PT$-ring, with square root axiom [1, Corollary 6.2]; quote Theorem 3.

It results from the corollary that if two normal elements are similar via an unbounded element, they are already similar via a bounded (even unitary) element; in particular, a normal bounded element cannot be similar to a normal unbounded element. Normality is essential here, as is shown by the following example due to Jacob Feldman:

Example 6. Let $A$ be the $C^*$-sum of denumerably many copies of the algebra $K_2$ of $2 \times 2$ complex matrices. $A$ may be represented as the algebra of all functions $n \mapsto f(n)$ ($n = 1, 2, 3, \ldots$), with $f(n) \in K_2$, $\|f(n)\|$ bounded, and operations pointwise. Since $K_2$ is its own regular ring, the regular ring $C$ of $A$ is the algebra of all functions $n \mapsto f(n)$ with $f(n) \in K_2$. Consider the functions $f$, $g$, $h \in C$ defined by

$$f(n) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad g(n) = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}, \quad h(n) = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}.$$ 

Since $h(n)f(n)h(n)^{-1} = g(n)$ for all $n$, one has $hfh^{-1} = g$. Thus $f$ and $g$ are similar in $C$, even though $f$ is bounded (i.e., is an element of $A$) and $g$ is not bounded.

2. More on the regular ring. Throughout, $C$ denotes the regular ring of a finite $AW^*$-algebra $A$ (of unrestricted type).

If $x \in C$, and $RP(x) = 1$, then $x$ is invertible. For, $Cx = C$ [1, Corollary 7.1], so there exists $y \in C$ with $yx = 1$; moreover $LP(x) \sim RP(x) = 1$, hence $LP(x) = 1$ by finiteness, $xC = C, xz = 1$ for suitable $z$. Note that an $x \in C$ is invertible if and only if it is left (right) invertible.

If $x \in C$ is invertible, then $x^*$ is invertible, and $(x^*)^{-1} = (x^{-1})^*$; if moreover $x$ is self-adjoint, so is $x^{-1}$. 


Lemma 1. If $x \in C$, $x \geq 0$, and $x$ is invertible, then $x^{-1} \geq 0$.

Proof. Say $xy = yx = 1$, and $x = z^*z$ [1, Definition 6.1]. Then $(yz^*)z = 1$ shows that $z$ is invertible (see above remarks), hence $x^{-1} = (z^*z)^{-1} = (z^{-1})(z^{-1})^* \geq 0$.

Lemma 2. Let $a \in A$, $0 \leq a \leq 1$, and suppose $a$ has an inverse in $C$. Then $a^{-1} \geq 1$.

Proof. Say $ax = xa = 1$; we know $x \geq 0$ from Lemma 1. Write $x = y^2$, $y$ self-adjoint [1, Corollary 6.2]. Since $(ay)y = y(ya) = 1$, $y$ is invertible, and $ay = ya = y^{-1}$. Then $a \leq 1$, $y^*ay \leq y^*y$, $yay \leq y^2$, $ay^2 \leq y^2$, $ax \leq x$, $1 \leq x$.

Theorem 6. Suppose $x, y \in C$, $0 \leq x \leq y$, and $x$ is invertible. Then $y$ is invertible, and $x^{-1} \geq y^{-1} \geq 0$.

Proof. The relation $0 \leq x \leq y$ implies $RP(x) \leq RP(y)$ (see the proof of [1, Corollary 7.6]); by assumption $RP(x) = 1$, hence $RP(y) = 1$, $y$ is invertible. Write $x^{1/2} = wy^{1/2}$ with $w \in A$, $w^*w \leq 1$ [1, Corollary 7.6]. Then $w = (x^{1/2})(y^{1/2})^{-1}$ is invertible in $C$, hence so is $w^*w$, and $(w^*w)^{-1} = (w^{-1})(w^{-1})^* \geq 1$ by Lemma 2. Since $(x^{1/2})^{-1} = (y^{1/2})^{-1}w^{-1}$, one has $x^{-1} = (x^{1/2})^{-1}(x^{1/2})^{-1} = (y^{1/2})^{-1}(w^{-1})(w^{-1})^*(y^{1/2})^{-1} \geq (y^{1/2})^{-1} \cdot (y^{1/2})^{-1} = y^{-1}$. For a similar result of Rellich, see [4, Hilfsatz 4].

Corollary. Suppose $A$ has the property that every increasingly directed family of self-adjoint elements, which is bounded above, has a least upper bound. Then $C$ has the same property.

Proof. For ease of notation, we write the proof for sequences. Suppose $x_i \in C$ are self-adjoint, $x_1 \leq x_2 \leq x_3 \leq \cdots$, $y \in C$ is self-adjoint, and $x_i \leq y$ for all $i$. Adding $-x_1$ throughout, we can assume $0 \leq x_i \leq y$. Then $1 \leq 1 + x_1 \leq 1 + x_2 \leq \cdots \leq 1 + y$, hence by Theorem 6, $1 \geq (1 + x_1)^{-1} \geq (1 + x_2)^{-1} \geq \cdots \geq (1 + y)^{-1} \geq 0$. But $(1 + x_i)^{-1}$ and $(1 + y)^{-1}$ belong to $A$ [1, Lemma 5.1]. Let $a \in A$ be the greatest lower bound of the $(1 + x_i)^{-1}$; one has $0 \leq (1 + y)^{-1} \leq a \leq (1 + x_i)^{-1}$. By Theorem 6, $a$ has an inverse in $C$, and $1 + y \geq a^{-1} \geq 1 + x_i$. Evidently $a^{-1} - 1$ is a least upper bound for the $x_i$. (Example: $A$ any finite $W^*$-algebra; see [8, Theorem 1].)

Lemma. Let $z \in C$ be normal, and suppose there exists a complex number $\lambda$ such that $z - \lambda$ has an inverse in $A$. Then the relations $a \in A$, $az = za$, imply $az^* = z^*a$.

Proof. (We are assuming, so to speak, that the "resolvent set" of $z$ is nonempty.) Suppose $a \in A$, $az = za$. Then $a(z - \lambda) = (z - \lambda)a$, and $z - \lambda$ is normal. Changing notation, assume $z$ invertible, $z^{-1} \in A$, .
az = za. Then $z^{-1}a = az^{-1}$, hence by Fuglede's theorem $a(z^{-1})^* = (z^{-1})^*a$, $a(z^*)^{-1} = (z^*)^{-1}a$, $z^*a = az^*$.

**Theorem 7.** Let $z \in C$ be normal, and write $z = x + iy$ with $x$ and $y$ self-adjoint. Suppose there exists a real number $\alpha$ such that $x - \alpha$ (or $y - \alpha$) has an inverse in $A$. Then the relations $a \in A$, $az = za$, imply $az^* = z^*a$.

**Proof.** Passing to $iz$ if necessary, we may suppose that it is $x - \alpha$ which has a bounded inverse. Then $(x - \alpha)^{-2} = (x - \alpha)^{-1}(x - \alpha)^{-1} \leq \beta$ for a suitable real number $\beta > 0$. By Theorem 6, $(x - \alpha)^2 \geq 1/\beta > 0$. Since $yx = xy$ by normality, and $z - \alpha = (x - \alpha) + iy$, we have $(z - \alpha)^* \cdot (z - \alpha) = (x - \alpha)^2 + y^2 \geq (x - \alpha)^2 \geq 1/\beta > 0$. Hence $(z - \alpha)^*(z - \alpha)$ is invertible, and $(z - \alpha)^{-1}(z - \alpha)^{-1} \leq \beta$. Therefore $(z - \alpha)^{-1} \in A$ [1, Lemma 5.1]; quote the lemma.

A self-adjoint $x \in C$ is semi-bounded in case there exists a real number $\beta$ such that either $x \leq \beta$ or $x \geq \beta$. For instance if $x \in A$ is self-adjoint, then $x \leq \|x\|$. If $x$ is semi-bounded, say $x \geq \beta$, then setting $\alpha = \beta - 1$, one has $x - \alpha \leq 1$, hence $x - \alpha$ has a bounded inverse (Theorem 6, and Lemma 5.1 of [1]). Thus:

**Corollary.** Let $z \in C$ be normal, and write $z = x + iy$, with $x$ and $y$ self-adjoint. Suppose either $x$ or $y$ is semi-bounded. Then the relations $a \in A$, $az = za$, imply $az^* = z^*a$.

If $A$ has a trace (e.g. if $A$ is Type I, or is a finite $W^*$-algebra), it is clear that the relations $a \in A$, $a^*a \leq aa^*$, imply $a^*a = aa^*$. We do not know if every finite $AW^*$-algebra $A$ has this property, but whenever $A$ does, so does $C$:

**Theorem 8.** Suppose the relations $a \in A$, $a^*a \leq aa^*$, imply $a^*a = aa^*$. Then the relations $x \in C$, $x^*x \leq xx^*$, imply $x^*x = xx^*$.

**Proof.** Suppose $x^*x \leq xx^*$. Write $x = ur$, $r \geq 0$, $u$ unitary [1, Corollary 7.4]. Then $x^*x = r^2$, and $xx^* = ur^2u^* = u(x^*x)u^*$. Setting $s = x^*x$, $t = xx^*$, we have $0 \leq s \leq t$, and $s$, $t$ are unitarily equivalent. Set $b = (1 + s)^{-1}$, $c = (1 + t)^{-1}$; clearly $b$, $c$ are unitarily equivalent, in fact $usu^* = t$ yields $ubu^* = c$. Moreover $b \geq c$ by Theorem 6, and $b$, $c \in A$ [1, Lemma 5.1]. Set $a = b^{1/2}u^*$. Then $aa^* = b^{1/2}u^*ub^{1/2} = b \geq c = ubu^* = a^*a$. By the hypothesis on $A$, $aa^* = a^*a$, hence $b = c$, and this leads to $s = t$.

**Corollary 1.** Suppose the relations $a \in A$, $a^*a \leq aa^*$, imply $a^*a = aa^*$. If $x \in C$, and $x$ commutes with $x^*x$, then $x$ is normal.

**Proof.** By assumption $xx^*x = x^*xx$. Right-multiplying by $x^*$,
\[ xx*xx* = x*x xx* \] Setting \( r = x*x, s = xx* \), we have \( r \geq 0, s \geq 0 \), and \( s^2 = rs \). In particular \( rs \) is self-adjoint, so that \( rs = sr \). Hence by uniqueness of positive square roots, \( s = (s^2)^{1/2} = (rs)^{1/2} = r^{1/2}s^{1/2} \) (see \([1, \text{remarks following Definition 6.3}]\)). Then \( 0 \leq (r^{1/2} - s^{1/2})^2 = r - 2r^{1/2}s^{1/2} + s = r - 2s + s = r - s \), thus \( 0 \leq s \leq r \). That is, \( xx* \leq x*x \), hence \( xx* = x*x \) by Theorem 8.

**Remark.** In an infinite algebra \( B \), choose \( x \in B \) with \( x^*x = 1 \) but \( xx* \neq 1 \). Then \( x \) commutes with \( x^*x \), but is not normal.

**Corollary 2.** Suppose the relations \( a \in A, a^*a \leq aa^* \), imply \( a^*a = aa^* \). Then every triangular normal matrix in \( C_n \) is diagonal.

**Proof.** Suppose e.g. \( n = 3 \), and

\[
\begin{pmatrix}
 a & b & c \\
 0 & d & e \\
 0 & 0 & f
\end{pmatrix}
\]

is a normal element of \( C_3 \). From the 1-1 position in the relation \( z^*z = zz^* \), one has \( a^*a = aa^* + bb^* + cc^* \), thus \( a^*a - aa^* = bb^* + cc^* \geq 0 \) \([1, \text{Theorem 6.1}]\), \( aa^* \leq a^*a \). By Theorem 8, \( aa^* = a^*a \), hence \( b = c = 0 \) \([1, \text{Lemma 3.4}]\). Inspection now of the 2-2 position similarly yields \( e = 0 \). The case for general \( n \) is an obvious induction.

**Remark.** If \( B \) is an infinite algebra, there exists a normal (even unitary) matrix

\[
\begin{pmatrix}
 a & b \\
 0 & c
\end{pmatrix}
\]

in \( B_2 \) with \( b \neq 0 \). For, choose a partial isometry \( a \in A \) with \( a^*a = 1, 
aa^* = e \neq 1 \), and set \( b = 1 - e, c = a^* \).

**Addenda.** (1) I am indebted to J. Dixmier for calling my attention to the references \([4] \) and \([8] \).

(2) Recently M. Rosenblum has given a beautiful proof of the Fuglede-Putnam theorem; for bounded operators, the proof is non-spatial (see \([10] \)).

**References**


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**A CORRECTION AND IMPROVEMENT OF A THEOREM ON ORDERED GROUPS**

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In this note the notation and terminology of [1] will be used throughout. In particular, $G$ will always denote an $o$-group with well ordered rank. Let $P$ be the multiplicative group of positive rational numbers, and let $R$ be the additive group of real numbers. In [1] the proofs of Theorems 2 and 3 are incorrect. This is a result of the careless formulation of the theorems by the author. Consider the following properties of $G$.

1. Each component $G^\gamma/G^\gamma$ of $G$ has its group of $o$-automorphisms isomorphic to a subgroup $P^\gamma$ of $P$.
2. Each component $G^\gamma/G^\gamma$ of $G$ is $o$-isomorphic to a subgroup $D^\gamma$ of $R$, and the only $o$-automorphisms of $D^\gamma$ are multiplications by some elements of $P$.
3. For each pair $\alpha \in \alpha$ and $\gamma \in \Gamma$, there exists a pair $m, n$ of positive integers such that $ng\alpha \equiv mg \mod G^\gamma$ for all $g$ in $G^\gamma$.

The statements of Theorems 2 and 3 include the hypothesis (1), but (2) and (3) are actually used in the proofs. Clearly (2) implies (1).

**Lemma.** (a) (2) is independent of the particular choice of $D^\gamma$. (b) (2) implies (3). (c) (1) does not imply (2) or (3).

**Proof.** (a) Let $\sigma$ be an $o$-isomorphism of the subgroup $A$ of $R$ onto the subgroup $B$ of $R$, and suppose that the only $o$-automorphisms of $A$ are multiplications by some elements of $P$. If $\beta$ is an $o$-automor-

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