CHARACTERIZATION OF SOME ELEMENTARY TRANSFORMATIONS

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1. **Introduction.** The purpose of this paper is to study some elementary transformations of surfaces embedded in a 3-dimensional Euclidean space E^3. This will be developed analogously to the following theorem [1]:

TRANSLATION THEOREM. *Given two closed orientable surfaces* S, $\bar{S}$ *and a homeomorphism* $h: S \to \bar{S}$ *such that*: (1) *each line joining corresponding points is parallel to a fixed direction* E, (2) *the mean curvatures at corresponding points are equal*; *moreover* S, $\bar{S}$ *are assumed not to contain pieces of cylinders in E-direction. Then* h *is a translation.*

All surfaces mentioned will be of class C^2. The notations in [1] will be adopted except that German letters will be replaced by corresponding capital English ones. For example: X, N, H and dA are respectively used to denote the position vector, the unit vector along inward normal direction, the mean curvature and the surface element of a surface S. When a second surface $\bar{S}$ is mentioned, the corresponding quantities are represented by the same letters with bars above them. As in [1], the following formulas will be used:

$$dX \times dX = 2NdA, \tag{1.1}$$

$$dX \times dN = -2HNdA. \tag{1.2}$$

A closed nonself-intersecting surface S is said to be convex with respect to a given point 0, (1) if every straight line through 0 meets S at no point, at one point of contact or at two distinct points, (2) if there is a differentiable homeomorphism $f: S \to S$ such that each straight line joining corresponding points passes through 0.

We intend to prove the following theorems:

THEOREM 1. *Given two closed orientable surfaces* S, $\bar{S}$ *and a differentiable homeomorphism* $h: S \to \bar{S}$ *such that*: (1) *each straight line* $P\bar{P}$ *joining the corresponding points* P *and* $\bar{P}$ *passes through a fixed point* 0; (2) *with* 0 *as origin, the quantities* X, $\bar{X}$, H, $\bar{H}$ *are related to each other either by* (i) $\bar{H}\bar{X} = HX$ *throughout* S *and* $\bar{S}$ *or by* (ii) $\bar{H}\bar{X} = -HX$ *throughout* S *and* $\bar{S}$. *Moreover* S, $\bar{S}$ *are assumed not to contain pieces of cones with vertex* 0. *Then* h *is a homothetic transformation with center* 0 *and with a*

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positive or negative constant of proportionality according as (i) *or* (ii) *holds.*

THEOREM 2. *Given two closed orientable surfaces* S, $\bar{S}$ *and a differentiable homeomorphism* $h: S\to\bar{S}$, *such that*: (1) *each segment* $P\bar{P}$ *joining the corresponding points* P *and* $\bar{P}$ *subtends a constant angle* $P0\bar{P}$ *about a fixed point* 0, (2) *with* 0 *as origin,* HX *and* $\bar{H}\bar{X}$ *are equal in magnitude. Moreover,* S, $\bar{S}$ *are assumed not to contain pieces of cones with vertex* 0. *Then* h *is a similarity with* 0 *as center of similitude.*

THEOREM 3. *Given two closed orientable surfaces* S, $\bar{S}$ *and a differentiable homeomorphism* $h: S\to\bar{S}$ *such that*: (1) *each straight line* $P\bar{P}$ *joining corresponding points* P *and* $\bar{P}$ *passes through a fixed point* 0; (2) *with* 0 *as origin, the quantities* X, $\bar{X}$, H, $\bar{H}$ *are related to each other either by* (i) $\bar{H}\bar{X}=-(H+2X\cdot N/X\cdot X)X$ *throughout* S *and* $\bar{S}$; *or by* (ii) $\bar{H}\bar{X}=(H+2X\cdot N/X\cdot X)X$ *throughout* S *and* $\bar{S}$. *Moreover* S, $\bar{S}$ *are assumed not to contain either pieces of cones with vertex* 0 *or the point* 0 *itself. Then* h *is an inversion with center* 0 *and with real or pure imaginary radius of inversion according as* (i) *or* (ii) *holds.*

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2. Proofs of theorems.

PROOF OF THEOREM 1. Assume $\bar{H}\bar{X}=HX$. Write $\bar{X}=kX$, where $k=H/\bar{H}$.

CASE 1. $0\notin S$ and $0\notin\bar{S}$. Then $k\neq 0,\ \infty$.

$$\begin{aligned} d\bar{X}\times d\bar{X} &= (kdX+Xdk)\times(kdX+Xdk)\\ &= k^2(dX\times dX)+2k(Xdk\times dX). \end{aligned}$$

By (1.1), we have

$$\bar{N}d\bar{A} = k^2NdA + k(Xdk\times dX),$$

whose scalar product with $\bar{H}\bar{X}(=HX)$ gives

$$(\bar{X}\cdot\bar{N})\bar{H}d\bar{A} = k^2(X\cdot N)HdA. \tag{2.1}$$

Let $a=(N\times X)\cdot dX$, $b=(\bar{N}\times X)\cdot dX$ and note that $ddX=0$, then by use of (1.1) and (1.2), we obtain

$$\begin{aligned} da &= 2(X\cdot N)HdA+2dA,\\ db &= -X\cdot(d\bar{N}\times dX)+2(\bar{N}\cdot N)dA\\ &= (2/k^2)(\bar{X}\cdot\bar{N})\bar{H}d\bar{A}+2(\bar{N}\cdot N)dA \end{aligned}$$

since

$$\overline{X}\cdot(d\overline{N}\times d\overline{X}) = k^2 X\cdot(d\overline{N}\times dX).$$

Hence, by (2.1) we have

$$d(a-b)/2 = (1-\overline{N}\cdot N)dA. \tag{2.2}$$

From Stokes' Theorem, it is evident that

$$\iint_S (1-\overline{N}\cdot N)dA = 0.$$

Since $1-\overline{N}\cdot N\geqq 0$, and dA always keeps the same sign, we have

$$1-\overline{N}\cdot N = 0$$

and therefore

$$N = \overline{N}.$$

Moreover, since $\overline{N}\cdot d\overline{X}=0$, $N\cdot dX=0$, we have

$$(N\cdot X)dk = 0.$$

Hence $k=$constant, unless $N\cdot X=0$.

Let R be the set of points of S at which $N\cdot X=0$. Then every point of R (if there is any) is not an interior point; for otherwise, S would contain a piece of cone with vertex 0. Hence every point of R is a limiting point of $S-R$, and, due to the continuity of $k=H/\overline{H}$, $k=$constant throughout S. Moreover $N=\overline{N}$ implies that k is positive. Consequently h is a homothetic transformation with center 0 and with positive constant of proportionality.

CASE 2. $0\in S$ or $0\in\overline{S}$. Without loss of generality, we may assume $0\in S$. In any open set U of S containing 0, take a neighborhood V of 0. Let V' be the boundary of V (and so also of $S-V$). Since $(1-\overline{N}\cdot N)$ dA always keeps the same sign

$$\left|\iint_{S-U}(1-\overline{N}\cdot N)dA\right| \leqq \left|\iint_{S-V}(1-\overline{N}\cdot N)dA\right|. \tag{2.3}$$

The expression on the right of (2.3) is equal to

$$\frac{1}{2}\left|\int_{V'}[(N-\overline{N})\times X]\cdot dX\right|$$

because of (2.2) and Stokes' Theorem, and it can be made as small as we please by choosing V small enough, while the expression on the left of (2.3) remains fixed. Hence

$$\iint_{S-U}(1-\overline{N}\cdot N)dA = 0.$$

Following the same argument as in Case 1, we have k = positive constant in $S-U$ for every open set U of S containing 0. Hence k = positive constant throughout S, since k is continuous.

Assume $\overline{H}\overline{X}=-HX$. Write $\overline{X}=-kX$ where $k=H/\overline{H}$. Through similar arguments as above, we obtain

$$\iint_S (1+\overline{N}\cdot N)dA = 0,$$

which gives $\overline{N}=-N$ and therefore k = positive constant.

REMARK 1. Theorem 1 still holds when S and $\overline{S}$ are not closed but bounded with boundaries B and $\overline{B}$, such that $h(B)=\overline{B}$ and at corresponding points on B and $\overline{B}$, we have $\overline{N}=N$ for case (i) or $\overline{N}=-N$ for case (ii). This is evident, because

$$\iint_S (1-\overline{N}\cdot N)dA = \frac{1}{2}\int_B [(N-\overline{N})\times X]\cdot dX \qquad \text{for case (i)},$$

$$\iint_S (1+\overline{N}\cdot N)dA = \frac{1}{2}\int_B [(\overline{N}+N)\times X]\cdot dX \qquad \text{for case (ii)}.$$

REMARK 2. If we consider the more general condition $HX=r\overline{H}\overline{X}$, where r is a constant, without loss of generality, we may assume $|r|\leqq 1$. Then instead of (2.2) we get

$$\frac{1}{2}d(a-rb) = (1-rN\cdot\overline{N})dA.$$

Hence

$$\iint_S (1-r\overline{N}\cdot N)dA = 0.$$

This equation implies

$$1-rN\cdot\overline{N} = 0$$

which is impossible unless $r=\pm 1$.

COROLLARY. *Given a closed orientable surface S convex with respect to a fixed point* 0. *With* 0 *as origin, the quantities X, H, X', H' at points corresponding under f are related to each other by $H'X'=-HX$. Then S is symmetric with respect to* 0.

PROOF. It is clear that f: $S\rightarrow S$ satisfies the assumptions in Theorem 1. Hence it is a homothetic transformation with center 0 and with negative constant of proportionality $-k$. Since both PP' and $P'(P')'$

pass through 0, $(P')'$ should be either P or P', and since f is one-one, $(P')'=P$. Hence $k^2=1$, and $k=1$. Therefore S is symmetric with respect to 0.

PROOF OF THEOREM 2. There is a transformation g in E^3 (which is either a single rotation about an axis through 0 or such a rotation followed by a reflection against a plane through 0), such that each straight line $0P$ is transformed into $0\overline{P}$ where P, $\overline{P}$ are points corresponding under h.

Let $S^*=g(S)$. It is clear that hg^{-1}: $S^*\to\overline{S}$ satisfies the assumptions of Theorem 1, and hence is a homothetic transformation with center 0. Therefore h is a similarity.

PROOF OF THEOREM 3. Let g be the inversion about the unit sphere with center 0. Denote by X^*, H^* and N^* the position vector, the mean curvature and the unit vector along inward normal direction at $P^*=g(P)$ of $S^*=g(S)$, respectively. By simple calculations we obtain

$$H^*X^* = -\left(H + 2\frac{X\cdot N}{X\cdot X}\right)X,$$

which reduces to $H^*X^*=\overline{H}\overline{X}$ for case (i) and to $H^*X^*=-HX$ for case (ii). Hence hg^{-1}: $S^*\to\overline{S}$ is a homothetic transformation with center 0 and with positive or negative constant of proportionality according as (i) or (ii) holds. Thus $h=(hg^{-1})g$ is an inversion about 0, and the radius of inversion is real or pure imaginary according as (i) or (ii) holds.

REMARK. Theorem 3 still holds when S and $\overline{S}$ are not closed but bounded with boundaries B and $\overline{B}$ such that $h(B)=\overline{B}$ and at corresponding points on the boundaries $\overline{N}=-N+2(X\cdot N/X\cdot X)X$ for case (i) or $\overline{N}=N-2(X\cdot N/X\cdot X)X$ for case (ii). This is evident because $N^*=-N+2(X\cdot N/X\cdot X)X$.

COROLLARY. *If S is a closed orientable surface convex with respect to a point* 0 *not on S, and with* 0 *as origin we have $H=-(X\cdot N/X\cdot X)X$. Then S is a sphere with center* 0.

PROOF. Since $HX=-(H+2(X\cdot N/X\cdot X))X$, each point of S is invariant under the inversion about a sphere with center 0 and with real radius. Consequently, S itself is a sphere with center 0.

REFERENCE

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