ENTIRE OPERATORS AND FUNCTIONAL EQUATIONS

Z. A. MELZAK

1. It is our aim to study some equations of the form

$$\frac{\partial f(x, t)}{\partial t} = K[f(x, t)], \tag{1}$$

where $0 \leqq x, t < \infty$, $f(x, 0)$ is known and $K(f)$ has a representation

$$K(f) = \sum_{1}^{\infty} K_n(f), \tag{2}$$

each $K_n(f)$ being derived from an n-linear operator by equating the argument functions.

Equations of type (1) occur in physics, especially in connection with transport phenomena and kinetic theory. In this context the summands $K_n(f)$ in (2) possess a simple interpretation: for $n \geqq 2$ $K_n(f)$ is the contribution toward the time-rate of reshuffling of a distribution $f(x, t)$, due to n-tuple collisions, n-fold coalescences or similar processes, while $K_1(f)$ is the contribution due to breakup or other destructive process. For further details about this, and other background material, see [**1**; **2**] and [**3**].

Our main result is a theorem asserting unique local existence of solutions of Equation (1). These are obtained by the Picard iterative method and bounds on various approximation errors are given or can be developed. However, these estimates seem to be too crude for actual computation.

2. Let I be the interval $[0, \infty)$ and let C^0, L^1 and L^∞ denote respectively the classes of continuous, integrable and essentially bounded functions on I. Let $\mathfrak{F} = C^0 \cap L^1 \cap L^\infty$ and let $\| \ \|_1$ and $\| \ \|_\infty$ be the L^1 and L^∞ norms. For each positive integer n let K_n be a mapping satisfying the following conditions:

$$K_n\colon \mathfrak{F} \times \mathfrak{F} \times \cdots \times \mathfrak{F} \ (n \text{ times}) \to C^0, \tag{3}$$

(4) for each x $K_n(f_1, \cdots, f_n)$ is a symmetric n-linear operator,

$$\|K_n(f_1, \cdots, f_n)\|_1 \leqq C_n \prod_{1}^{n} \|f_i\|_1, \tag{5}$$

Presented to the Society August 29, 1958; received by the editors May 21, 1958 and, in revised form, October 16, 1958.

$$\|K_1(f)\|_\infty \leqq D_1(\|f\|_1 + \|f\|_\infty),$$

$$(6) \qquad \|K_n(f_1, \cdots, f_n)\|_\infty \leqq D_n \prod_1^n \|f_i\|_1 \sum_1^n \|f_i\|_\infty / \|f_i\|_1, \qquad n \geqq 2.$$

Here C_n and D_n are non-negative constants. Although this terminology is not quite justified, K_n will be called an operator rather than a mapping. Define now $K_n(f) = K_n(f, \cdots, f)$ and introduce the functions

$$F(x) = \sum_1^\infty C_n x^n, \qquad G(x) = \sum_1^\infty D_n x^n;$$

if both are entire in x, the operator $K(f) = \sum_1^\infty K_n(f)$ will be called an entire operator. It will be noticed that the norm-functions F and G are then absolutely monotonic over I. The following example shows that the class of entire operators has nontrivial members. Let $\psi(x, y)$ be a non-negative, bounded, continuous function, defined for $0 \leqq y \leqq x$, and let

$$\int_0^x y\psi(x, y)dy \leqq x, \qquad \int_0^x \psi(x, y)dy \leqq E < \infty.$$

For each $n \geqq 2$ let $\phi_n(x_1, \cdots, x_n)$ be a non-negative, bounded, continuous function, invariant under any permutation of its arguments. Let

$$K_1(f) = \int_x^\infty f(y)\psi(y, x)dy - f(x)/x \int_0^x y\,\psi(x, y)\,dy,$$

$$\begin{aligned} &K_n(f_1, \cdots, f_n) \\ &= 1/a_n \Big\{ \int_0^x \int_0^{x-x_1} \cdots \int_0^{x-x_1-\cdots-x_{n-2}} P_n(x - x_1 - \cdots - x_{n-1}) \\ &\qquad \cdot \phi_n(x_1, \cdots, x_{n-1}, x - x_1 - \cdots - x_{n-1}) dx_{n-1} \cdots dx_1 \\ &\quad - \sum_{j=1}^n \int_0^\infty \cdots \int_0^\infty P_j(x)\phi_n(x_1, \cdots, x_{j-1}, x, x_{j+1}, \cdots x_n) \\ &\qquad dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_n \Big\}, \end{aligned}$$

where we have put for brevity

$$\begin{aligned} P_j(u) &= P_j(u, x_1, \cdots, x_n) \\ &= f_1(x_1) \cdots f_{j-1}(x_{j-1}) f_j(u) f_{j+1}(x_{j+1}) \cdots f_n(x_n). \end{aligned}$$

Under these conditions $K(f)$ is entire if the positive constants a_n tend to infinity sufficiently fast.

3. Let $f, g \in \mathfrak{F}$. In this section estimates will be derived for the L' and L^∞ norms of $K(f)-K(g)$, K being an entire operator. By the n-linearity and symmetry of K_n we have

$$
\begin{aligned}
K_n(f) - K_n(g) &= K_n(f-g, f, \cdots, f) \\
&\quad + K_n(f-g, g, f, \cdots, f) + \cdots + K_n(f-g, g, \cdots, g).
\end{aligned}
\tag{7}
$$

Taking L' norms and making use of (5), we obtain

$$
\begin{aligned}
\|K_n(f) - K_n(g)\|_1 &\leqq C_n \|f-g\|_1 \sum_0^{n-1} \|f\|^i \|g\|^{n-1-i} \\
&= C_n \|g-f\|_1 \frac{\|f\|_1^n - \|g\|_1^n}{\|f\|_1 - \|g\|_1}.
\end{aligned}
\tag{8}
$$

Summing over n and recalling the definition and properties of $F(x)$, we get

$$
\|K(f) - K(g)\|_1 \leqq \sum_1^\infty \|K_n(f) - K_n(g)\|_1 \leqq \|f-g\|_1 \frac{F(\|f\|_1) - F(\|g\|_1)}{\|f\|_1 - \|g\|_1}
$$

and finally,

$$
\|K(f) - K(g)\|_1 \leqq \|f-g\|_1 F'(\max(\|f\|_1, \|g\|_1)).
\tag{9}
$$

Similarly we obtain an L^∞ estimate. Starting again from (7), taking L^∞ norms and making use of (6), we have for $n \geqq 2$

$$
\begin{aligned}
\|K_n(f) - K_n(g)\|_\infty &\leqq D_n \|f-g\|_\infty \sum_0^{n-1} \|f\|_1^i \|g\|_1^{n-1-i} + D_n \|f-g\|_1 \\
&\cdot \left\{ \|f\|_\infty \sum_0^{n-1} i \|f\|_1^{i-1} \|g\|_1^{n-1-i} + \|g\|_\infty \sum_0^{n-1} (n-1-i) \|f\|_1^i \|g\|_1^{n-1-i-1} \right\}.
\end{aligned}
$$

The series occurring in the above are easily summed up, and

$$
\begin{aligned}
\|K_n(f) - K_n(g)\|_\infty &\leqq D_n \|f-g\|_\infty \frac{\|f\|_1^n - \|g\|_1^n}{\|f\|_1 - \|g\|_1} + D_n \|f-g\|_1 \\
&\cdot \left\{ \|f\|_\infty \left[\frac{n \|f\|_1^{n-1}}{\|f\|_1 - \|g\|_1} - \frac{\|f\|_1^n - \|g\|_1^n}{(\|f\|_1 - \|g\|_1)^2} \right] \right. \\
&\left. + \|g\|_\infty \left[\frac{n \|g\|_1^{n-1}}{\|g\|_1 - \|f\|_1} - \frac{\|f\|_1^n - \|g\|_1^n}{(\|f\|_1 - \|g\|_1)^2} \right] \right\}.
\end{aligned}
\tag{10}
$$

By the definition and properties of $G(x)$ we have

$$\begin{aligned}\sum_1^\infty D_n\left[\frac{nu^{n-1}}{u-v}-\frac{u^n-v^n}{(u-v)^2}\right] &= \frac{G'(u)}{u-v}-\frac{G(u)-G(v)}{(u-v)^2} \\ &= \frac{G(v)-G(u)-(v-u)G'(u)}{(v-u)^2},\end{aligned}\tag{11}$$

and by a mean-value theorem

$$\begin{aligned}&\frac{G(v)-G(u)-(v-u)G'(u)}{(v-u)^2} \\ &\qquad = 1/2G''[u+\theta(v-u)] \leqq 1/2G''(\max(u, v)),\end{aligned}\tag{12}$$

where $0\leqq\theta\leqq 1$. Now

$$\begin{aligned}\|K(f)-K(g)\|_\infty &\leqq \sum_1^\infty \|K_n(f)-K_n(g)\|_\infty \\ &\leqq D_1(\|f-g\|_1+\|f-g\|_\infty)+\sum_2^\infty D_n\|K_n(f)-K_n(g)\|_\infty;\end{aligned}$$

making use of (10), (11) and (12), we obtain finally

$$\begin{aligned}\|K(f)-K(g)\|_\infty \leqq \|f-g\|_\infty G'(\max(\|f\|_1, \|g\|_1)) + \|f-g\|_1 \\ \cdot\{D_1+1/2G''(\max(\|f\|_1, \|g\|_1))(\|f\|_\infty+\|g\|_\infty)\}.\end{aligned}\tag{13}$$

As a special case of estimates (9) and (13) we obtain

$$\|K(f)\|_1 \leqq F(\|f\|_1), \qquad \|K(f)\|_\infty \leqq D_1\|f\|_1+\|f\|_\infty G'(\|f\|_1).\tag{14}$$

The following lemma belongs logically to this section:

LEMMA 1. *If $f\in\mathfrak{F}$ and $K(f)$ is an entire operator, then $K(f)\in\mathfrak{F}$.*

It suffices to show that $K(f)\in C^0$, the rest follows from (14). By its definition and by hypothesis (3) $K(f)$ is an infinite series of continuous functions $K_n(f)$. Since $|K(f)|\leqq\|K(f)\|_\infty$, it follows from (6) that the series for $K(f)$ converges uniformly and absolutely, therefore $K(f)\in C^0$.

4. Consider the integrated form of Equation (1):

$$f(x, t) = f(x, 0) + \int_0^t K[f(x, \tau)]d\tau.\tag{15}$$

Let $f(x, 0)$ be the given initial value of $f(x, t)$ and let $f(x, 0)\in\mathfrak{F}$. Define now the Picard sequence of successive approximations, $\{f_j\}$, by

$$f_0 = f(x, 0), \qquad f_{j+1} = f_0 + \int_0^t K(f_j)d\tau;\tag{16}$$

here, and elsewhere, the arguments x, t or x, τ will be left out unless explicitly necessary. Let

$$(17)\qquad A_j = A_j(t) = \|f_j\|_1, \qquad B_j = B_j(t) = \|f_j\|_\infty.$$

It is necessary to show first that the sequence $\{f_j\}$ is defined, that is, that $f_j \in \mathfrak{F}$ for all j.

LEMMA 2. *There exists $t_1 > 0$, such that on the interval $[0, t_1]$ both A_j and B_j are uniformly bounded.*

Taking L' norms in (16) we have, by Fubini's theorem and by (14),

$$(18)\qquad A_{j+1} \leqq A_0 + \int_0^t F(A_j)d\tau.$$

Let $x_1 > A_0$ and put

$$(19)\qquad M = F'(x_1), \quad t_1 = 1/M \log x_1/A_0, \quad x_1 = A_0 e^{Mt_1}.$$

Since F' is nondecreasing, $F(x) \leqq Mx$ for $x \in [0, x_1]$. Therefore

$$A_1 \leqq A_0 + A_0 M \int_0^t e^{M\tau} d\tau = A_0 e^{Mt} \leqq x_1, \quad 0 \leqq t \leqq t_1,$$

and by induction on j in (18),

$$(20)\qquad A_j \leqq A_0 e^{Mt}, \qquad 0 \leqq t \leqq t_1.$$

B_j can be bounded similarly. We observe first that

$$(21)\qquad \underset{x}{\text{l.u.b.}} \left| \int_0^t h(x, \tau) d\tau \right| \leqq \int_0^t \underset{x}{\text{l.u.b.}} \left| h(x, \tau) \right| d\tau;$$

now, taking L^∞ norms in (16), we have by (14) and (21)

$$B_{j+1} \leqq B_0 + \int_0^t [D_1 A_j + G'(A_j)B_j] d\tau.$$

In view of (20) this implies that

$$(22)\quad B_{j+1} \leqq B_0 + D_1 A_0/M(e^{Mt} - 1) + G'(A_0 e^{Mt}) \int_0^t B_j d\tau, \ 0 \leqq t \leqq t_1,$$

and by an easy induction on j,

$$(23)\qquad B_j \leqq \{B_0 + D_1 A_0/M(e^{Mt} - 1)\} e^{c_1 t}, \qquad 0 \leqq t \leqq t_1,$$

where $c_1 = G'(A_0 \exp Mt_1)$.

This completes the proof of Lemma 2. It follows now from Lemmas 1 and 2 that $\{f_j\} \subset \mathfrak{F}$ for $0 \leqq t \leqq t_1$. From now on, until the contrary is mentioned, it will be assumed that the variable t, wherever it occurs, is restricted to the interval $[0, t_1]$.

LEMMA 3. *The series*

$$\sum_0^\infty \|f_{j+1} - f_j\|_1, \qquad \sum_0^\infty \|f_{j+1} - f_j\|_\infty$$

are uniformly, and of course absolutely, convergent.

We have from (16)

$$f_{j+1} - f_j = \int_0^t \{K(f_j) - K(f_{j-1})\} d\tau; \tag{24}$$

taking L' norms, interchanging the order of integrations and using (9), we obtain

$$\|f_{j+1} - f_j\|_1 \leqq \int_0^t \|f_j - f_{j-1}\|_1 F'(\max (A_j, A_{j-1}))d\tau;$$

therefore by (20)

$$\|f_{j+1} - f_j\|_1 \leqq c_2 \int_0^t \|f_j - f_{j-1}\|_1 d\tau, \qquad c_2 = F'(A_0 e^{Mt_1}).$$

By induction on j we show now that

$$\|f_{j+1} - f_j\|_1 \leqq c_2 (c_3 t)^j / j! \tag{25}$$

where

$$\|f_1 - f_0\| \leqq A_0 + A_1 \leqq 2A_0 e^{Mt_1} = 2c_3.$$

Similarly, by the estimate (13),

$$\begin{aligned} \|K(f_j) - K(f_{j-1})\|_\infty &\leqq \|f_j - f_{j-1}\|_\infty G'(\max (A_j, A_{j-1})) \\ &\quad + \|f_j - f_{j-1}\|_1 \{D_1 + 1/2G''(\max (A_j, A_{j-1}))(\|f_j\|_\infty + \|f_{j-1}\|_\infty)\}. \end{aligned} \tag{26}$$

Let

$$B_0 + D_1 A_0 / M(e^{Mt_1} - 1)e^{c_1 t_1} = c_4, \qquad D_1 + c_4 G''(c_3) = c_5; \tag{27}$$

then (26) and (23) imply that

$$\|K(f_j) - K(f_{j-1})\|_\infty \leqq c_1 \|f_j - f_{j-1}\|_\infty + c_5 \|f_j - f_{j-1}\|_1. \tag{28}$$

Now, by taking L^∞ norms in (24) and using (21) and (28), we get

$$\|f_{j+1} - f_j\|_\infty \leqq \int_0^t [c_1 \|f_j - f_{j-1}\|_\infty + c_5 \|f_j - f_{j-1}\|_1] d\tau,$$

which implies, in view of (25), that

$$(29)\qquad \|f_{j+1}-f_j\|_\infty \leqq c_6\frac{(c_3t)^j}{j!}+c_1\int_0^t\|f_j-f_{j-1}\|_\infty d\tau,\qquad c_6=c_2c_5/c_3.$$

Let

$$\Delta_j=\|f_{j+1}-f_j\|_\infty,\qquad E_j=c_6(c_3t_1)^j/j!;$$

then, by (23),

$$\Delta_0=\|f_1-f_0\|_\infty\leqq\|f_0\|_\infty+\|f_1\|_\infty\leqq B_0+c_4=c_7,$$

so that (29) can be written as

$$(30)\qquad \Delta_j\leqq E_j+c_1\int_0^t\Delta_{j-1}d\tau,\qquad \Delta_0\leqq c_7.$$

By induction on j in (30) we obtain

$$\Delta_j\leqq c_7(c_1t)^j/j!+\sum_{k=1}^{j}E_k(c_1t)^{j-k}/(j-k)!.$$

By the definition of E_j, and by the above estimate, we obtain finally

$$(31)\qquad \begin{aligned}\Delta_j&\leqq c_7(c_1t)^j/j!+c_6\sum_{k=1}^{j}(c_3t_1)^k/k!(c_1t)^{j-k}/(j-k)!\\&\leqq c_7(c_1t_1)^j/j!+c_6(c_3t_1+c_1t_1)^j/j!.\end{aligned}$$

Now the lemma follows from the estimates (25) and (31).

5. THEOREM 1. *Let K be an entire operator and let $f(x,0)$ be a given member of $\mathfrak{F}$. Then Equation* (1) *either possesses a unique solution $f(x,t)$ in $\mathfrak{F}$, valid for $t\geqq 0$, or else, it possesses a unique solution $f(x,t)\in\mathfrak{F}$ on an interval $[0,T)$ and $\|f(x,t)\|_1$ tends to infinity as t approaches T.*

By Lemma 3 the series

$$f(x,0)+\sum_0^\infty[f_{j+1}(x,t)-f_j(x,t)],$$

majorized by the absolutely and uniformly convergent series

$$f(x,0)+\sum_0^\infty\|f_{j+1}(x,t)-f_j(x,t)\|_\infty,$$

defines a continuous function $f(x,t)\in\mathfrak{F}$ which satisfies Equation (14) and consequently Equation (1), on the interval $[0,t_1]$. Suppose that there are two such solutions, $f(x, t)$ and $g(x, t)$, both in $\mathfrak{F}$ and let $f(x, 0)=g(x, 0)$. Then

$$f - g = \int_0^t \{K(f) - K(g)\} d\tau,$$

and by (9)

$$\|f - g\|_1 \leqq c \int_0^t \|f - g\|_1 d\tau,$$

where c is a constant. But this implies that $\|f-g\|_1=0$, and since f and g are continuous, $f\equiv g$. This establishes the unique solution $f(x, t)$ on $[0, t_1]$. However, $f(x, t_1)\in\mathfrak{F}$ and so we may continue the solution successively to $[t_1, t_2]$, $[t_2, t_3]$ etc. Let $A(t)=\|f(x, t)\|_1$ and suppose that the extension process stops: $t_n\to T<\infty$. If $A(t)$ is uniformly bounded from above on the whole of $[0, T)$, then it is possible to construct an existence interval $[T-\epsilon, T+\eta]$ for some $\epsilon,\eta>0$. With suitable changes, this can be shown in the same way as in (19). This leads to a contradiction and the theorem is proved.

6. THEOREM 2. *The solution f of Equation* (1), *obtained in Theorem* 1, *is of the class C^∞ in t, for every x.*

Since f satisfies Equation (1) and $K(f)\in\mathfrak{F}$, f is in C^1 in t. Differentiating the series

$$\frac{\partial f}{\partial t} = \sum_1^\infty K_n(f)$$

and using the symmetry and n-linearity of K_n, we obtain formally

$$\frac{\partial^2 f}{\partial t^2} = \sum_1^\infty \sum_1^\infty nK_n[K_m(f), f, \cdots, f], \tag{32}$$

and in the same way

$$\begin{aligned} \frac{\partial^3 f}{\partial t^3} = \sum_1^\infty \sum_1^\infty \sum_1^\infty \{&n(n-1)K_n[K_p(f), K_m(f), f, \cdots, f] \\ &+ nmK_n[K_m(K_p(f), f, \cdots, f) f, \cdots, f]\}. \end{aligned} \tag{33}$$

In each case the right hand side may be used to define the left hand one, and the second and third derivatives will exist if only the series in (32) and (33) can be shown to converge uniformly and absolutely. Since the general expression for the Nth derivative is rather involved, we shall introduce a symbolic procedure to simplify the matters. Define an operation ∂ by

$$\partial K = K/(1 - K) = \sum_1^\infty K^n,$$

let $\partial^2 K = \partial(\partial K)$ be

$$\partial^2 K = \sum_1^\infty \partial K^n = \sum_1^\infty nK^{n-1}\partial K = \sum_1^\infty \sum_1^\infty nK^{n-1}K^m,$$

in the same way

$$\begin{aligned}\partial^3 K &= \sum_1^\infty \sum_1^\infty n\partial(K^{n-1}K^m) = \sum_1^\infty \sum_1^\infty [n\partial K^{n-1}K^m + nK^{n-1}\partial K^m] \\ &= \sum_1^\infty \sum_1^\infty [n(n-1)K^{n-2}\partial K K^m + nmK^{n-1}K^{m-1}\partial K] \\ &= \sum_1^\infty \sum_1^\infty \sum_1^\infty [n(n-1)K^{n-2}K^pK^m + nmK^{n-1}K^{m-1}K^p],\end{aligned}$$

and so on. If now K^n is interpreted as $K_n(f)$, $K^{n-1}K^m$ as $K_n[K_m(f), \cdots, f]$, $K^{n-2}K^pK^m$ as $K_n[K_p(f), K_m(f), f, \cdots, f]$, and $K^{n-1}K^{m-1}K^p$ as $K_n\{K_m[K_p(f), f, \cdots, f], f, \cdots, f\}$ etc., we obtain a compact notation for the derivatives: $\partial^N f/\partial t^N = \partial^N K$. When written out at length, this becomes a sum of a finite number of expressions of the following form:

$$S_N = \sum_{n_1=i_1}^\infty \cdots \sum_{n_N=i_N}^\infty n_1!/(n_1 - i_1)! \cdots n_N!/(n_N - i_N)! \; K^{n_1-i_1} \cdots K^{n_N-i_N}, \tag{34}$$

where $\{i_k\}$ is any system of non-negative integers for which $i_N = 0$ and $\sum_1^N i_k = N-1$. When the symbolic powers of K are finally interpreted as the proper iterates of the K_n operators, we obtain the Nth derivative itself. It remains to show that the N-tuply infinite series in (34), each term of which stands for a continuous function, will converge uniformly and absolutely. We have $|K(f)| \leqq ||K(f)||_\infty$, and by the main estimates (5), (6) and (14) it is easy to show that

$$||K^{n_1-i_1} \cdots K^{n_N-i_N}||_1 \leqq \left(\prod_{k=1}^N C_{n_k}\right) ||f||_1^{n_1+\cdots+n_N-(N-1)},$$

$$||K^{n_1-i_1} \cdots K^{n_N-i_N}||_\infty \leqq \left(\sum_{k=1}^N P_k H_{kn_1} \cdots H_{kn_N}\right) ||f||_1^{n_1+\cdots n_N-N} ||f||_\infty, \tag{35}$$

where H_{kn_s} is either C_{n_s} or D_{n_s} and P_k is a polynomial in $n_1, \cdots, n_N$ of degree $N-1$. These formulas show that the series in (34) is dominated by an absolutely and uniformly convergent series so that the N-th derivative $\partial^N f/\partial t^N$ exists for every N. Moreover, this derivative is of class L^1 in x on I.

7. In this last section we give without proof two theorems concerning the behaviour of the solutions of Equation (1) and those of a related equation. These theorems are generalizations of various results of [1] and [2] and proofs can be constructed without difficulty by slight adaptations of the methods of [1] and [2].

THEOREM 3. *If $C_n = D_n = 0$ for $n \geqq 3$, then the solution $f(x, t)$ of Equation* (1), *obtained in Theorem* 1, *is analytic in t for $t \in [0, T_1)$, $0 < T_1 \leqq T$, and it vanishes identically for $t \in [T_1, T)$.*

An entire operator $K(f)$ is called positive if $K(f) \geqq 0$ for $f \geqq 0$. An entire operator is called separable if it is a difference of two positive operators: $K(f) = K_+(f) - K_-(f)$. An entire operator is called conservative if the first moment $\int_0^\infty x K_n(f) dx$ exists for all n and if $\int_0^\infty x K(f) dx \equiv 0$. Since

$$\int_0^\infty x K(\lambda f) dx = \sum_1^\infty \lambda^n \int_0^\infty x K_n(f) dx \qquad \lambda > 0$$

it follows that if $K(f)$ is a conservative operator, then each summand $K_n(f)$ is also conservative. The example in §2 shows that there exist nontrivial separable conservative operators. A separable operator $K(f)$ is called a C-operator if there are constants c_1 and c_2, such that $K_-(f) \leqq c_1 f$ and $\int_0^\infty K(f) dx \leqq c_2 \int_0^\infty f dx$ for any $f \in \mathfrak{F}$, $f \geqq 0$.

THEOREM 4. *Let $K(t, f)$ be a t-dependent conservative C-operator and let all the conditions of boundedness, continuity, separability, etc., hold uniformly in t. Let $f(x, 0) \in \mathfrak{F}$ be non-negative with a finite first moment over I. Then the equation*

$$\frac{\partial f(x, t)}{\partial t} = K[t, f(x, t)]$$

possesses a unique non-negative solution $f(x, t) \in \mathfrak{F}$, valid for $t \geqq 0$.

REFERENCES

1. Z. A. Melzak, *A scalar transport equation*, Trans. Amer. Math. Soc. vol. 85, no. 2 (1957) pp. 547–560.

2. ———, *A scalar transport equation* II, Michigan Math. J. vol. 4 (1957) pp. 193–206.

3. D. Morgenstern, *Analytical studies related to the Maxwell-Boltzmann equation*, J. Rat. Mech. Anal. vol. 4 (1955).

McGILL UNIVERSITY