SIMPLICITY OF NEAR-RINGS OF TRANSFORMATIONS
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1. Introduction. Consider a group $\mathfrak{G} = (G, +)$ (not necessarily commutative) and $T(G)$, the set of transformations on $G$. Define addition (+) and multiplication (·) on $T(G)$ by

$$(1.1) \quad g(A + B) = gA + gB, \quad g(AB) = (gA)B, \quad g \in G, A, B \in T(G).$$

Then $(T(G), +, \cdot) = \mathfrak{T}(\mathfrak{G})$ is a near-ring, the near-ring of transformations on $\mathfrak{G}$. That is (i) $(T(G), +)$ is a group, (ii) $(T(G), \cdot)$ is a semigroup, and (iii) multiplication is left distributive with respect to addition:

$$(1.2) \quad A(B + C) = AB + AC, \quad A, B, C \in T(G).$$

The transformation $0$, where $g0 = 0$, for all $g \in G$, is the zero of $\mathfrak{T}(\mathfrak{G})$. Let $T_0(G)$ be the set of all transformations which commute with the zero transformation, i.e. $0A = 0, A \in T(G)$. $T_0(G)$ determines a sub-near-ring $\mathfrak{T}_0(\mathfrak{G})$ of $\mathfrak{T}(\mathfrak{G})$.

The main theorem is now stated.

**Theorem 1.** For any group $\mathfrak{G}$, $\mathfrak{T}(\mathfrak{G})$ and $\mathfrak{T}_0(\mathfrak{G})$ are simple.

That is, they have no proper nontrivial homomorphic images.

2. Preliminaries. A subset $Q$ of a near-ring $\mathfrak{B}$ determines an ideal of $\mathfrak{B}$ if and only if

(a) $(Q, +)$ is a normal subgroup of $(P, +)$,
(b) $PQ \subset Q$,
(c) $(a + q)b - ab$ is in $Q$ for all $a, b \in P, q \in Q$.

As in ring theory, the kernal $\mathfrak{Q}$ of a homomorphism $\theta$ from a near-ring $\mathfrak{B}$ to a near-ring $\mathfrak{B}'$ (i.e. the inverse image of the zero of $P'$) is an ideal. Every ideal $\mathfrak{Q}$ is the kernal of the natural homomorphism $\nu: a\nu = a + Q$, from $\mathfrak{B}$ to the difference near-ring $\mathfrak{B} - \mathfrak{Q}$, and every homomorphic image $\mathfrak{B}/\mathfrak{Q}$ with kernel $\mathfrak{Q}$ is isomorphic to $\mathfrak{B} - \mathfrak{Q}$. Thus a near-ring $\mathfrak{B}$ is simple if and only if its only ideals are itself and the zero ideal.

By way of warning, the following three concepts are introduced. A subset $Q$ of a near-ring $\mathfrak{B}$ determines (α) a left ideal if it satisfies (a) and (b), (β) a right ideal if it satisfies (a) and (b') $QP \subset Q$, (γ) a two-sided ideal if it satisfies (a), (b) and (b'). While an ideal is a left ideal, examples show that ideals need not be two-sided ideals, and

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that two-sided ideals need not be ideals.

In the near-ring $\mathfrak{T}(\mathcal{G})$, denote by $Z_g$ the transformation such that $g'Z_g = g$, $g' \in G$. Clearly $AZ_g = Z_g$, and $Z_gA = Z_{gA}$, for every $A$ in $T(G)$ and $g$ in $G$. In particular $Z_0 = 0$ in $(T(G), +)$.

**Lemma 1.** If $\mathfrak{Q}$ is an ideal of $\mathfrak{T}(\mathcal{G})$, then $QT_0(G) \subseteq \mathfrak{Q}$.

From (c), letting $a \in Q$, $b \in T_0(G)$, $(Z_0 + a)b - Z_0b = ab \in Q$. In particular, if $\mathfrak{Q}$ is an ideal of $\mathfrak{T}_0(\mathcal{G})$, then $\mathfrak{Q}$ is a two-sided ideal of $\mathfrak{T}_0(\mathcal{G})$.

**Lemma 2.** The only sub-near-ring of $\mathfrak{T}(\mathcal{G})$ which contains $\mathfrak{T}_0(\mathcal{G})$ properly is $\mathfrak{T}(\mathcal{G})$ itself.

Let $\mathfrak{B}$ be a sub-near-ring of $\mathfrak{T}(\mathcal{G})$ containing $\mathfrak{T}_0(\mathcal{G})$. Consider $A \in \mathfrak{B}$, $A \in T_0(G)$. Then $A - Z_{0A} \in T_0(G)$. Hence $Z_{0A} \in \mathfrak{B}$. Since $0A \neq 0$, and $Z_{0A}B = Z_{0(AB)}$, and $T_0(G)$ is transitive on the nonzero elements of $G$, the set of $Z_g$'s, $g \in G$ is in $P$. Thus, $C \in T(G)$, $C = (C - Z_{0C}) + Z_{0C}$ is in $P$, for $C - Z_{0C}$ is in $T_0(G)$.

**3. Proof of the theorem.** A transformation $A \in T(G)$ has rank $R(A) = R$ if the set $\{gA \mid g \in G\}$ has cardinality $R$.

**Lemma 3.** A nonzero ideal $\mathfrak{I} \subseteq \mathfrak{T}_0(\mathcal{G})$ contains all the elements of rank 2.

By Lemma 1, $\mathfrak{I}$ is a two-sided ideal, and since $\mathfrak{I} \neq \{0\}$, there exists a $V \in \mathfrak{I}$ and $g_1, g_1' \in G$, $g_1' \neq 0$ such that $g_1V = g_1'$. Partition $G$ into disjoint sets $G_1$ and $G_2$, $0 \in G_2$. Define $A \in T_0(G)$ such that $gA = g_1$, $g \in G_1$, $gA = 0$, $g \in G_2$. Let $g''$ be any element of $G$, and let $B \in T_0(G)$ be such that $g_1'B = g''$. Then $gA VB = g''$, $g \in G_1$, $gA VB = 0$, $g \in G_2$, and $A VB \in \mathfrak{I}$.

**Lemma 4.** If $\mathcal{G}$ is finite, $\mathfrak{T}_0(\mathcal{G})$ is simple.

Suppose $I$ contains all elements of rank less than or equal to $k$. Partition $G$ into pairwise disjoint nonempty sets $G_0, G_1, \ldots, G_k$, $0 \in G_0$. Consider $k + 1$ elements in $G$, $g_0 = 0$, $g_1$, $\ldots$, $g_k$. Define $A \in I$ such that $gA = g_i$, $g \in G_i$, $i = 0, 1, \ldots, k - 1$; $gA = 0$, $g \in G_k$. Define $B \in T_0(G)$ such that $gB = 0$, $g \in G_i$, $i = 0, 1, \ldots, k - 1$, $gB = g_k$, $g \in G_k$. Hence $C = A + B \in I$ and has rank $k + 1$. Since the sets $G_i$, $i = 0, 1, \ldots, k$, and the elements $g_i$, $i = 1, 2, \ldots, k$ are arbitrary, $I = T_0(G)$ by induction.

**Lemma 5.** Let $\mathcal{G}$ have infinite cardinality and let $h \in G$, $h \neq 0$. Then there exists a maximal set $A \subseteq G$ such that $A \cap (A + h) = \emptyset$. Further, $A \cup (A + h) \cup (A - h) = G$. Hence the cardinality of $A$, $A + h$, and $A - h$ are each equal to the cardinality of $G$. 


Consider the collection of subsets of $G$: $S = \{S \mid (S+h) \cap S = \emptyset \}$. The collection $S$ is not empty since $\{0\} \in S$. Define a partial ordering $S_1 > S_2$ if $S_1 \supset S_2$, and $S_1, S_2 \in S$. Consider a linearly ordered subcollection $\{S_t \mid t \in T, S_t \in S\}$. Then, it is asserted that $S' = \bigcup_{t \in T} S_t \in S$ and $S' > S_t, t \in T$. Trivially, $S' \supset S_t, t \in T$. Suppose $s' = s'' + h, s', s'' \in S'$. But, $s', s'' \in S_t$ for some $t$, a contradiction. Hence by Zorn's lemma a maximal set $A$ exists.

Let $k \in G, k \in A, k \in A + h$. (If no such element $k$ exists, then $G = A \cup (A + h)$ and the sets $A$ and $A + h$ each have the cardinality of $G$.) The elements $k + h = a \in A$. For if not, consider $A' = \{A, k\}$. Then $A' + h$ is disjoint from $A'$, contrary to the maximality of $A$. Therefore $k = a - h \in A - h$. Since $A, A + h, A - h$ have the same cardinality and their union is $G$, the lemma is proved.

**Lemma 6.** If $I$ contains a transformation of rank $d$, then $I$ contains every transformation of rank less than or equal to $d$.

Let $\{G_x \in G \mid x \in X\}$ be any partition of $G$ into pairwise disjoint sets, where $X$ is an index set of cardinality $d$, with $0 \in \mathbb{N}$. Consider any collection $\{g'_x \in G \mid x \in X, g'_x = 0\}$. Let $V \in I$ have rank $d$ and denote the elements in the image $GV$ of $V$ by $\{g_x \mid x \in X, g_x = 0\}$. For each $g_x \in GV$, let $g''_x$ be an element such that $g''_x V = g_x$. Define $A \in T_0(G)$ such that $gA = g''_x, g \in G, x \in X$. Let $B$ be any element in $T_0(G)$ such that $g_x B = g'_x$. Then $A VB$ is an arbitrary transformation of rank less than or equal to $d$ and is in $I$.

**Lemma 7.** If $\emptyset$ has infinite cardinality $\aleph_0(\emptyset)$ is simple.

Define the transformation $D_h \in T_0(G)$ by $gD_h = h, g \neq 0, g \in G$. Then by Lemma 3, $D_h \in I$. Define $C \in T_0(G)$ by $gC = g, g \in A; gC = 0, g \in A$, where $A$ is a maximal set (Lemma 5) such that $A \cap (A + h) = \emptyset$. Then $T = (1 + D_h)C - C \in I$, where $1$ is the identity map. Observe that $gT = (g + h)C - gC = -g, g \in A$. Hence $T$ has rank of the same cardinality as $G$. Thus, by the previous lemma, $I = T_0(G)$. It is only in Lemma 7 that the invariance property of an ideal is used in proving the simplicity of $\mathfrak{T}_0(\emptyset)$.

**Lemma 8.** $\mathfrak{T}(\emptyset)$ is simple.

If $G$ has order 2, then the theorem is easily checked directly. Assume therefore that $G$ has order greater than 2. If $\mathfrak{J}$ is a nonzero ideal in $\mathfrak{T}(\emptyset)$, and if there exists a $C \in I \cap T_0(G), C \neq 0$, then $T_0(G) \subseteq I$ by Lemmas 1, 4, 7. In addition, since an ideal is a left ideal, $Z_0C = Z_0C \subseteq I, g \in G$. Choose $g \in G$ so that $gC \neq 0$. Then, since the smallest near-ring properly containing $\mathfrak{T}_0(\emptyset)$ is $\mathfrak{T}(\emptyset)$, the lemma follows. Finally if the ideal contains no nonzero element of $T_0(G)$, consider $B \neq 0 \in I$ and
g \in G \text{ such that } gB = g_1 \neq 0. \text{ Then } Z_0B = Z_{g_1} \subseteq I. \text{ Let } C \in T_0(G) \text{ such that } g_1C = 0 \text{ and } (g_2 + g_1)C \neq g_2C \text{ for some } g_2 \in G, \ g_2 \neq g_1, \ g_2 \neq 0. \text{ Then } D = (1 + Z_0)C - C \subseteq I. \text{ Further } D \in T_0(G) \text{ and } g_2D \neq 0, \text{ contrary to the assumption.}

The theorem follows from Lemmas 4, 7, 8.

4. Two-sided invariant sub-near-rings. A sub-near-ring \( \mathfrak{Q} \) of a near-ring \( \mathfrak{P} \) is two-sided invariant if conditions (b) and (b') of §2 hold.

**Lemma 9.** The sets (i) \( T_r(G) = \{ A \in T(G) | R(A) = 1 \} \), (ii) \( T_f(G) = \{ A \in T(G) | R(A) < \aleph_0 \} \), (iii) \( T_{\aleph_k}(G) = \{ A \in T(G) | R(A) \leq \aleph_k \} \), where \( \aleph_k \) is an infinite cardinal number, determine two-sided invariant sub-near-rings of \( T(\mathfrak{G}) \). Further \( T_{\aleph_{k_1}}(G) \supseteq T_{\aleph_{k_2}}(G) \) provided \( \aleph_{k_1} > \aleph_{k_2} \), where \( \aleph \) is the cardinality of \( T(G) \).

The proof of this lemma is straightforward and depends on very simple properties of cardinal numbers. The theorem which follows shows that these near-rings determine the two-sided invariant sub-near-rings of \( T(\mathfrak{G}) \) and \( T_0(\mathfrak{G}) \).

**Theorem 2.** (i) The two-sided invariant sub-near-rings of \( T(\mathfrak{G}) \) are \( T_z(\mathfrak{G}) \), \( T_f(\mathfrak{G}) \), and \( T_{\aleph_k}(\mathfrak{G}) \) for any infinite cardinal number \( \aleph_k \) less than or equal to the cardinality of \( \mathfrak{G} \). (ii) The two-sided invariant sub-near-rings of \( T_0(\mathfrak{G}) \) are \( \{ 0 \} \) and the intersection of the two-sided invariant sub-near-rings of \( T(\mathfrak{G}) \) with \( T_0(\mathfrak{G}) \).

Let \( \mathfrak{P} \) be a two-sided invariant sub-near-ring of \( T(\mathfrak{G}) \) which has an element \( C \) of maximal infinite rank \( \aleph_k \). Then, for suitable choices of \( A, B \in T(G) \), \( ACB \) represents any transformation whose rank is less than or equal to \( \aleph_k \). (See proof of Lemma 6.) Thus \( \mathfrak{P} = T_{\aleph_k}(\mathfrak{G}) \).

If \( P \) contains no element of infinite rank, suppose \( P \) contains an element \( C \) of rank greater than 1. Then as above it follows that every element of rank less than or equal to the rank of \( C \) is in \( P \). The argument used in the proof of Lemma 4 is now valid to show that \( \mathfrak{P} = T_f(\mathfrak{G}) \).

References


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