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**QUASI-NIL RINGS**

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Rings have been studied which have among others the property that every commutator $xy - yx$ is in the nucleus. It seems appropriate to consider rings in which the square of every element is in the nucleus, a property that is shared by both associative and Lie rings. Under the additional assumptions of primeness and characteristic different from 2 it can be shown that such rings are either associative or have the property that $x^2 = 0$, for every element $x$ of the ring. If further $(x, y, z) + (y, z, x) + (z, x, y)$ is in the nucleus for all elements $x, y, z$ of the ring, then the ring is either associative or a Lie ring.

We use the notation $(x, y, z) = (xy)z - x(yz)$. The nucleus $N$ of a ring $R$ consists of all $n \in R$ such that $(n, R, R) = (R, n, R) = (R, R, n) = 0$. $N$ is a subring of $R$.

**Lemma.** Let $R$ be a prime ring satisfying $x^2 \in N$ for every $x \in R$ and of characteristic different from 2. Then either $R$ is associative or $N^2 = 0$.

**Proof.** For all $r, s \in R$, $rs + sr = (r + s)^2 - r^2 - s^2$ must be in $N$. Select $n, n' \in N$, and $x, y, z \in R$. Then $(n(n'x + xn'), y, z) = 0$, so that $(nn'x, y, z) = -(nxn', y, z)$. Similarly $(nxn', y, z) = -(nn'x, y, z)$ and $(xnn', y, z) = -(nn'x, y, z)$. By combining these three equalities it follows that $2(nn'x, y, z) = 0$. Assuming characteristic not 2 it then follows that $(nn'x, y, z) = 0$. Since $(nx, y, z) = ((nx)y)z - (nx)(yz) = (n(xy))z - n(x(yz)) = n((xy)z - n(x(yz)) = n(x, y, z)$, we replace $n$

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by \(nn'\) and obtain \((nn'x, y, z) = nn'(x, y, z)\). We deduce that \(nn'(x, y, z) = 0\), so that \(N^2(R, R, R) = 0\). Let \(I\) be the ideal generated by \(N^2\), and \(J\) the ideal generated by all associators \((R, R, R)\). We note that \(nn'x = n(n'x + xn') - (nx + xn)n' + xnn'\), so that \(N^2R \subseteq RN^2 + N^2\). Consequently \(I = RN^2 + N^2\). In an arbitrary ring \(J = (R, R, R) + (R, R, R)\). It follows from \(N^2(R, R, R) = 0\), that \(IJ = 0\). Since \(R\) is prime either \(I = 0\), or \(J = 0\). If \(I = 0\), then \(N^2 = 0\). On the other hand if \(J = 0\), then \(R\) is associative. This completes the proof of the lemma.

**Theorem 1.** Let \(R\) be a prime ring of characteristic not 2 satisfying \(r^2 \in N\) for all \(r \in R\). Then either \(R\) is associative or \(r^2 = 0\), for all \(r \in R\).

**Proof.** Let us consider the case \(N \neq R\). Then it follows from the lemma that \(N^2 = 0\). Let \(K = N + NR\). Since \(rn = (rn + nr) - nr \in K\) and \(snr = (sn + ns)r - nsr \in K\), for all \(n \in N\) and \(r, s \in R\), \(K\) must be an ideal of \(R\). Moreover if \(n' \in N\), \(nn' = n(rn' + n'r) - nn'r = 0\), since \(N^2 = 0\). This suffices to show \(K^2 = 0\). But \(R\) is prime and so \(K = 0\). But then \(N = 0\), whence \(r^2 = 0\), for all \(r \in R\). This completes the proof of the theorem.

**Theorem 2.** Let \(R\) be a prime ring of characteristic not 2 satisfying

(i) \(x^2 \in N\) for all \(x \in R\), and

(ii) \((x, y, z) + (y, z, x) + (z, x, y) \in N\) for all \(x, y, z \in R\). Then \(R\) is either associative or a Lie ring. Conversely all associative rings and all Lie rings satisfy both (i) and (ii).

**Proof.** Assume that \(R\) satisfies (i) and (ii), and suppose \(N \neq R\). Then Theorem 1 implies that \(x^2 = 0\), for all \(x \in R\), and consequently \(R\) is anti-commutative. For any \(n \in N\), \(nxy = -nxy = nxy\), and also \(nxy = - nxy\), so that \(2nxy = 0\). Thus \(NR^2 = 0\). The set \(T\) of all \(t \in R\) such that \(tR = 0\), forms an ideal of \(R\) which must be zero since \(R\) is prime. But since \(NR \subseteq T\) we obtain first \(NR = 0\), and subsequently \(N = 0\). In any anti-commutative ring \((x, y, z) + (y, z, x) + (z, x, y) = (xy)z - x(yz) + (yz)x - y(zx) + (zx)y - z(xy) = 2((xy)z + (yz)x + (zx)y)\) which equals twice the Jacobian of \(x, y, z\). Then because of (ii) we conclude that the well known Jacobi identity holds and thus \(R\) is a Lie ring. The converse follows automatically. In an associative ring \(N = R\), so that (i) and (ii) are trivially satisfied. All Lie rings are anti-commutative and satisfy the Jacobi identity, so that (ii) follows. This completes the proof of the theorem.

N. H. McCoy [1] and R. L. SanSoucie [2] have shown independently that any primitive ring is prime. Consequently the above theorems may be extended to primitive and semi-simple rings in the usual way.

We conclude with a couple of examples.
Example 1. Let \( x, y, z, n \) be basis elements of the algebra \( A \) over an arbitrary field \( F \). All products of basis elements are defined to be zero with the exception of \( xy = -yx = z \), \( zx = -xz = y \), \( yz = -zy = x \), and \( n^2 = n \). For every \( \alpha, \beta, \gamma, \delta \in F \), \( (\alpha x + \beta y + \gamma z + \delta n)^2 = \delta^2 n \) is in the nucleus. Thus for every \( r \in R \), \( r^2 \in N \). \( R \) has four ideals, the trivial ones, the ideal \( B \) generated by \( x, y, z \), and the ideal \( C \) generated by \( n \). Also \( BC = 0 = CB \), while \( B^2 = B \) and \( C^2 = C \).

Example 2. Let \( 1, x, y \) be basis elements of the algebra \( R \) over an arbitrary field \( F \), where \( xy = 1, yx = x^2 = y^2 = 0 \). For any \( \alpha, \beta, \gamma \in F \), \( (\alpha + \beta x + \gamma y)^2 = \alpha^2 + 2\alpha \beta x + 2\alpha \gamma y + \beta \gamma = 2\alpha (\alpha + \beta x + \gamma y) + \beta \gamma - \alpha^2 \). Thus \( R \) is quadratic over \( F \). Moreover it can be readily verified that \( R \) is simple, power-associative, and that all commutators of \( R \) are contained in \( F \). \( R \) is not associative since \( (x, y, y) = y \). Also \( (x + y)^2 = 1 \neq 0 \). If \( F \) happens to be a field of characteristic 2 then \( r^2 \in F \) for every \( r \in R \). We see that Theorem 1 fails to hold for rings of characteristic 2.

Bibliography


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