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QUASI-NIL RINGS

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Rings have been studied which have among others the property that every commutator $xy-yx$ is in the nucleus. It seems appropriate to consider rings in which the square of every element is in the nucleus, a property that is shared by both associative and Lie rings. Under the additional assumptions of primeness and characteristic different from 2 it can be shown that such rings are either associative or have the property that $x^2=0$, for every element x of the ring. If further $(x, y, z)+(y, z, x)+(z, x, y)$ is in the nucleus for all elements x, y, z of the ring, then the ring is either associative or a Lie ring.

We use the notation $(x, y, z)=(xy)z-x(yz)$. The nucleus N of a ring R consists of all $n\in R$ such that $(n, R, R)=(R, n, R)=(R, R, n)=0$. N is a subring of R.

Lemma. *Let R be a prime ring satisfying $x^2\in N$ for every $x\in R$ and of characteristic different from* 2. *Then either R is associative or $N^2=0$.*

Proof. For all $r, s\in R$, $rs+sr=(r+s)^2-r^2-s^2$ must be in N. Select $n, n'\in N$, and $x, y, z\in R$. Then $(n(n'x+xn'), y, z)=0$, so that $(nn'x, y, z)=-(nxn', y, z)$. Similarly $(nxn', y, z)=-(xnn', y, z)$ and $(xnn', y, z)=-(nn'x, y, z)$. By combining these three equalities it follows that $2(nn'x, y, z)=0$. Assuming characteristic not 2 it then follows that $(nn'x, y, z)=0$. Since $(nx, y, z)=((nx)y)z-(nx)(yz)=(n(xy))z-n(x(yz))=n((xy)z)-n(x(yz))=n(x, y, z)$, we replace n

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by nn' and obtain $(nn'x, y, z) = nn'(x, y, z)$. We deduce that $nn'(x, y, z) = 0$, so that $N^2(R, R, R) = 0$. Let I be the ideal generated by N^2, and J the ideal generated by all associators (R, R, R). We note that $nn'x = n(n'x + xn') - (nx + xn)n' + xnn'$, so that $N^2R \subset RN^2 + N^2$. Consequently $I = RN^2 + N^2$. In an arbitrary ring $J = (R, R, R) + (R, R, R)R$. It follows from $N^2(R, R, R) = 0$, that $IJ = 0$. Since R is prime either $I = 0$, or $J = 0$. If $I = 0$, then $N^2 = 0$. On the other hand if $J = 0$, then R is associative. This completes the proof of the lemma.

THEOREM 1. *Let R be a prime ring of characteristic not 2 satisfying $r^2 \in N$ for all $r \in R$. Then either R is associative or $r^2 = 0$, for all $r \in R$.*

PROOF. Let us consider the case $N \neq R$. Then it follows from the lemma that $N^2 = 0$. Let $K = N + NR$. Since $rn = (rn + nr) - nr \in K$ and $snr = (sn + ns)r - nsr \in K$, for all $n \in N$ and $r, s \in R$, K must be an ideal of R. Moreover if $n' \in N$, $nrn' = n(rn' + n'r) - nn'r = 0$, since $N^2 = 0$. This suffices to show $K^2 = 0$. But R is prime and so $K = 0$. But then $N = 0$, whence $r^2 = 0$, for all $r \in R$. This completes the proof of the theorem.

THEOREM 2. *Let R be a prime ring of characteristic not 2 satisfying* (i) *$x^2 \in N$ for all $x \in R$, and* (ii) *$(x, y, z) + (y, z, x) + (z, x, y) \in N$ for all $x, y, z \in R$. Then R is either associative or a Lie ring. Conversely all associative rings and all Lie rings satisfy both* (i) *and* (ii).

PROOF. Assume that R satisfies (i) and (ii), and suppose $N \neq R$. Then Theorem 1 implies that $x^2 = 0$, for all $x \in R$, and consequently R is anti-commutative. For any $n \in N$, $nxy = -xny = xyn$, and also $nxy = -xyn$, so that $2nxy = 0$. Thus $NR^2 = 0$. The set T of all $t \in R$ such that $tR = 0$, forms an ideal of R which must be zero since R is prime. But since $NR \subset T$ we obtain first $NR = 0$, and subsequently $N = 0$. In any anti-commutative ring $(x, y, z) + (y, z, x) + (z, x, y) = (xy)z - x(yz) + (yz)x - y(zx) + (zx)y - z(xy) = 2((xy)z + (yz)x + (zx)y)$ which equals twice the Jacobian of x, y, z. Then because of (ii) we conclude that the well known Jacobi identity holds and thus R is a Lie ring. The converse follows automatically. In an associative ring $N = R$, so that (i) and (ii) are trivially satisfied. All Lie rings are anti-commutative and satisfy the Jacobi identity, so that (ii) follows. This completes the proof of the theorem.

N. H. McCoy [1] and R. L. SanSoucie [2] have shown independently that any primitive ring is prime. Consequently the above theorems may be extended to primitive and semi-simple rings in the usual way.

We conclude with a couple of examples.

EXAMPLE 1. Let x, y, z, n be basis elements of the algebra A over an arbitrary field F. All products of basis elements are defined to be zero with the exception of $xy=-yx=z$, $zx=-xz=y$, $yz=-zy=x$, and $n^2=n$. For every α, β, γ, $\delta \in F$, $(\alpha x+\beta y+\gamma z+\delta n)^2=\delta^2 n$ is in the nucleus. Thus for every $r\in R$, $r^2\in N$. R has four ideals, the trivial ones, the ideal B generated by x, y, z, and the ideal C generated by n. Also $BC=0=CB$, while $B^2=B$, and $C^2=C$.

EXAMPLE 2. Let 1, x, y be basis elements of the algebra R over an arbitrary field F, where $xy=1$, $yx=x^2=y^2=0$. For any α, β, $\gamma \in F$, $(\alpha+\beta x+\gamma y)^2=\alpha^2+2\alpha\beta x+2\alpha\gamma y+\beta\gamma=2\alpha(\alpha+\beta x+\gamma y)+\beta\gamma-\alpha^2$. Thus R is quadratic over F. Moreover it can be readily verified that R is simple, power-associative, and that all commutators of R are contained in F. R is not associative since $(x, y, y)=y$. Also $(x+y)^2=1\neq 0$. If F happens to be a field of characteristic 2 then $r^2\in F$ for every $r\in R$. We see that Theorem 1 fails to hold for rings of characteristic 2.

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