THE ANALYSIS OF THE CHARACTERS OF THE LIE REPRESENTATIONS OF THE GENERAL LINEAR GROUP

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1. If a free Lie ring has n generators and A is a nonsingular $n \times n$ matrix of complex elements, then when the generators undergo a linear transformation of matrix A, the module of all forms of degree m in the generators is mapped into itself by a linear transformation of matrix $L_m(A)$ on a set of basis elements. The mapping $A \rightarrow L_m(A)$ is a representation of the full linear group known as the mth Lie representation [1]. The character of this representation has been shown [2] to be $\gamma_m = m^{-1}\sum_{d|m} \mu(d) s_d^{m/d}$, where $\mu(k)$ is the Möbius function of the integer k, and s_r is the sum of the rth powers of the eigenvalues of A. The decomposition of the mth Lie representation into its irreducible constituents is in exact correspondence with the analysis of the symmetric function γ_m into Schur functions. When m is prime there is a simple rule for the coefficient α_λ of any S-function $\{\lambda\}$ in γ_m, [2], but there is no such rule when m is composite. Since $s_d^{m/d} = \sum_\lambda \chi^\lambda_{d^{m/d}}\{\lambda\}$, where χ^λ_ρ is the irreducible character of the class (ρ) of the symmetric group $\mathfrak{S}_m$ corresponding to the partition (λ) of m, it follows that [10],

$$\alpha_\lambda = \frac{1}{m} \sum_{d|m} \mu(d) \chi^\lambda_{d^{m/d}}.$$

The calculation of α_λ thus reduces to the calculation of characters of the form $\chi^\lambda_{d^{m/d}}$.

It is the purpose of this note to suggest, in §2, a method of calculation of these characters, and, in §3, to indicate some relations satisfied by the α_λ.

2. The well known formulae for $\chi^\lambda_{1^m}$ are

$$\chi^\lambda_{1^m} = \frac{m! \prod_{r<s} (\lambda_r - \lambda_s - r + s)}{\prod_r (\lambda_r + p - r)!}$$

where p = number of parts in (λ);

$$\chi^\lambda_{1^m} = \frac{m!}{\prod_{r,s} (\lambda_r + \tilde{\lambda}_s - r - s + 1)},$$

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where $(\tilde\lambda)=(\tilde\lambda_1, \tilde\lambda_2, \cdots)$ is the partition conjugate to (λ);

$$\chi_{1^m}^{\lambda}=\frac{m!}{H_\lambda},$$

where H_λ is the product of the hook-lengths h_{ij} in the hook-graph of (λ) [**3**]. These give the coefficient of $\{\lambda\}$ in s_1^m. An alternative method, which extends to $\chi_{r^{m/r}}^{\lambda}$, the coefficient of $\{\lambda\}$ in $s_r^{m/r}$, is by evaluation of certain determinants; [**4**, pp. 134–135; **5**]. Thus to find $\chi_{1^{10}}^{531^2}$ we evaluate

$$10!\begin{vmatrix} \frac{1}{5!} & \frac{1}{6!} & \frac{1}{7!} & \frac{1}{8!} \\ \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \frac{1}{5!} \\ \cdot & 1 & \frac{1}{1!} & \frac{1}{2!} \\ \cdot & \cdot & 1 & \frac{1}{1!} \end{vmatrix}, \quad \text{or preferably} \quad 10!\begin{vmatrix} \frac{35}{8!} & \frac{4}{5!} \\ \frac{1}{5!} & \frac{1}{2!} \end{vmatrix},$$

obtained respectively from

$$\{531^2\}=\begin{vmatrix} \{5\} & \{6\} & \{7\} & \{8\} \\ \{2\} & \{3\} & \{4\} & \{5\} \\ \cdot & \{0\} & \{1\} & \{2\} \\ \cdot & \cdot & \{0\} & \{1\} \end{vmatrix} \quad \text{and} \quad \{531^2\}=\begin{vmatrix} \{51^3\} & \{21^3\} \\ \{5\} & \{2\} \end{vmatrix},$$

[**6**] giving 567. To find the coefficient of $\{531^2\}$ in $s_r^{10/r}$ we replace every $\{k\}$ in the first of these determinants by zero if k is not a multiple of r, and by $1/(k/r)!$ if it is. Multiplying the determinant by $(10/r)!$ gives the required coefficient. Thus the coefficients of $\{531^2\}$ in s_2^5, s_5^2, s_{10} are respectively

$$5!\begin{vmatrix} \cdot & \frac{1}{3!} & \cdot & \frac{1}{4!} \\ 1 & \cdot & \frac{1}{2!} & \cdot \\ \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{vmatrix} = 15, \qquad 2!\begin{vmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{vmatrix} = 2, \text{ and zero},$$

giving the coefficient of $\{531^2\}$ in γ_{10} as 55. This was given as 53 by Thrall [**1**], but was later corrected by Brandt [**2**].

Justification of the above procedure is given conveniently by the use of the differential operator D_λ obtained from $\{\lambda\}$ by replacing s_i^a by $i^a\partial^a/\partial s_i^a$, [7]. Thus the coefficient of $\{\lambda\}$ in s_1^m is $D_\lambda s_1^m$, and if $(\lambda)=(\lambda_1, \lambda_2, \cdots, \lambda_p)$, the coefficient can be written as

$$\begin{vmatrix} D_{\lambda_1} & D_{\lambda_1+1} & \cdots & D_{\lambda_1+p-1} \\ D_{\lambda_2-1} & D_{\lambda_2} & \cdots & D_{\lambda_2+p-2} \\ D_{\lambda_3-2} & D_{\lambda_3-1} & \cdots & D_{\lambda_3+p-3} \\ \cdot & \cdot & \cdot & \cdot \\ D_{\lambda_p-p+1} & & \cdots & D_{\lambda_p} \end{vmatrix} s_1^m$$

and the only effective part of each D_k is $(k!)^{-1}\partial^k/\partial s_1^k$. Then in $D_\lambda s_r^{m/r}$ the only effective part of each D_k is

$$\frac{1}{k!}\,\frac{k!}{r^{k/r}(k/r)!}\,r^{k/r}\,\frac{\partial^{k/r}}{\partial s_r^{k/r}}=\frac{1}{(k/r)!}\,\frac{\partial^{k/r}}{\partial s_r^{k/r}}$$

so that we get zero elements in the determinant when k is not a multiple of r, and elements $1/(k/r)!$ when k is a multiple of r.

3. We now list a number of results concerning the α_λ.

I. When m is prime, α_λ is the integer nearest to $m^{-1}\chi_{1^m}^\lambda$ [2].

II. $\alpha_m=0$ for $m>1$, [10], and $\alpha_{1^m}=0$ for $m>2$.

These follow since $\alpha_m=m^{-1}\sum\mu(d)$, and $\alpha_{1^m}=m^{-1}\sum(-1)^{m+m/d}\mu(d)$.

III. $\alpha_{m-1,1}=1$ for $m>1$, [10], and $\alpha_{21^{m-2}}=1$ for $m>2$.

These follow since

$$\chi_{1^m}^{m-1,1}=m-1,\quad\text{and}\quad\chi_{a^b}^{m-1,1}=-1\quad\text{for } a>1, b\geqq 1,$$

[8, p. 137] and so

$$\alpha_{m-1,1}=m^{-1}\left[m-1-\sum_{d\neq 1}\mu(d)\right]=1\qquad\text{for } m>1,$$

$$\alpha_{21^{m-2}}=m^{-1}\left[m-1-\sum_{d\neq 1}(-1)^{m+m/d}\mu(d)\right]=1,\qquad\text{for } m>2.$$

IV. If m is odd or a multiple of four, then $\alpha_\lambda=\alpha_{\tilde\lambda}$. It is well known that $\chi_{d^{m/d}}^\lambda=\chi_{d^{m/d}}^{\tilde\lambda}$ for odd values of d, and also for even d whenever m/d is even. This proves IV.

When m is twice an odd integer, $\chi_{d^{m/d}}^\lambda=-\chi_{d^{m/d}}^{\tilde\lambda}$ for even d, and in this case γ_m, expressed in power sums, can be written as $P+Q$, where the coefficients of $\{\lambda\}$ and $\{\tilde\lambda\}$ are the same in P, but have opposite signs in Q. P has all $s_d^{m/d}$ with odd d, and Q has all $s_d^{m/d}$ with even d. There are certain partitions (λ) for which the coefficient of $\{\lambda\}$ in Q is zero, and for these $\alpha_\lambda=\alpha_{\tilde\lambda}$.

The following three results are typical of many which can be obtained by equating appropriate coefficients when the right hand side of

$$m^{-1}\sum_{d|m}\mu(d)s_d^{m/d} = \sum_\lambda \alpha_\lambda\{\lambda\}$$

is expressed in terms of power sums by writing

$$\{\lambda\} = (m!)^{-1}\sum_\rho h_\rho \chi_\rho^\lambda S_\rho,$$

where h_ρ is the order of the class $(\rho)=1^{x_1}2^{x_2}\cdots$ of $\mathfrak{S}_m$, and $S_\rho = s_1^{x_1}s_2^{x_2}\cdots$.

V. $\sum_\lambda \chi_{1^m}^\lambda \alpha_\lambda = (m-1)!$ This is obtained by equating coefficients of s_1^m.

VI. $\sum_{\lambda=(m-r,1^r)}(-1)^r\alpha_\lambda = \mu(m)$. Obtained by equating coefficients of s_m.

VII. If S-functions of ranks one and two are written respectively in Frobenius notation as

$$\{\lambda\} = \begin{Bmatrix} X_r \\ Y_r \end{Bmatrix} \quad \text{and} \quad \{\nu\} = \begin{Bmatrix} X_{t_1} & X_{t_2} \\ Y_{t_1} & Y_{t_2} \end{Bmatrix},$$

and a, b are any two unequal positive integers such that $a+b=m$, then $\sum_\lambda \theta_r\alpha_\lambda + \sum_\nu \phi_t\alpha_\nu = 0$, where

$$\begin{aligned}
\theta_r &= (-1)^{Y_r} && \text{if } m > X_r \geqq a, \\
&= 0 && \text{if } a > X_r \geqq b, \\
&= (-1)^{Y_r+1} && \text{if } b > X_r \geqq 0, \\
\phi_t &= (-1)^{Y_{t_1}+Y_{t_2}} && \text{if } X_{t_1} + Y_{t_1} = a-1, \\
&= (-1)^{Y_{t_1}+Y_{t_2}+1} && \text{if } X_{t_1} + Y_{t_2} = a-1, \\
& && \text{or if } X_{t_2} + Y_{t_1} = a-1, \\
&= 0 \text{ otherwise.}
\end{aligned}$$

This result follows by equating coefficients of s_as_b, [9]. Other special relations of this kind may be obtained by equating coefficients of $s_as_bs_c$, and of other terms. They depend on a knowledge of expressions such as θ_r, ϕ_t above for the appropriate characteristics of $\mathfrak{S}_m$.

VIII. If h_d is the order of the class $(d^{m/d})$ of $\mathfrak{S}_m$, then

$$\sum_\lambda \alpha_\lambda^2 = \frac{(m-1)!}{m}\sum_d \frac{1}{h_d},$$

for square free values of d.

We have $D_{\gamma_m}\gamma_m = \sum_\lambda \alpha_\lambda D_\lambda \sum_\lambda \alpha_\lambda\{\lambda\} = \sum_\lambda \alpha_\lambda^2$. Also

$$\begin{aligned} D_{\gamma_m}\gamma_m &= m^{-1}\left[\sum_d \mu(d)d^{m/d}\frac{\partial^{m/d}}{\partial s_d^{m/d}}\right]m^{-1}\sum_d \mu(d)s_d^{m/d} \\ &= (m^2)^{-1}\left[\sum (\mu(d))^2 d^{m/d}(m/d)!\right] \\ &= (m^2)^{-1}\sum \frac{m!}{h_d} \end{aligned}$$

for square free values of d.

More explicit results may be written down when m has some prescribed form. Thus when m is a prime p, $\sum \alpha_\lambda^2 = p^{-1}[1+(p-1)!]$, and when m is the product of two distinct primes p, q,

$$\sum \alpha_\lambda^2 = \frac{1}{pq}\left[(pq-1)! + p^{q-1}(q-1)! + q^{p-1}(p-1)! + 1\right].$$

When m is not twice an odd number we have $\alpha_\lambda = \alpha_{\tilde{\lambda}}$ by IV. But if $m=2(2k+1)$, then by evaluating $D_{\gamma_m}\tilde{\gamma}_m$ in two ways, we obtain a further result;

IX. If m is twice an odd number, then $\sum_\lambda \alpha_\lambda\alpha_{\tilde{\lambda}} = (m-1)!/m \cdot \sum_d (-1)^{d+1}(h_d)^{-1}$ for square free values of d.

Some results on the characters of the Lie representations in the special case when the number of generators is two have been given in a recent paper by Davis [10].

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