

THE ANALYSIS OF THE CHARACTERS OF THE LIE REPRESENTATIONS OF THE GENERAL LINEAR GROUP

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1. If a free Lie ring has n generators and A is a nonsingular $n \times n$ matrix of complex elements, then when the generators undergo a linear transformation of matrix A , the module of all forms of degree m in the generators is mapped into itself by a linear transformation of matrix $L_m(A)$ on a set of basis elements. The mapping $A \rightarrow L_m(A)$ is a representation of the full linear group known as the m th Lie representation [1]. The character of this representation has been shown [2] to be $\gamma_m = m^{-1} \sum_{d|m} \mu(d) s_d^{m/d}$, where $\mu(k)$ is the Möbius function of the integer k , and s_r is the sum of the r th powers of the eigenvalues of A . The decomposition of the m th Lie representation into its irreducible constituents is in exact correspondence with the analysis of the symmetric function γ_m into Schur functions. When m is prime there is a simple rule for the coefficient α_λ of any S -function $\{\lambda\}$ in γ_m , [2], but there is no such rule when m is composite. Since $s_d^{m/d} = \sum_\lambda \chi_\lambda^{m/d} \{\lambda\}$, where χ_ρ^λ is the irreducible character of the class (ρ) of the symmetric group \mathfrak{S}_m corresponding to the partition (λ) of m , it follows that [10],

$$\alpha_\lambda = \frac{1}{m} \sum_{d|m} \mu(d) \chi_\lambda^{m/d}.$$

The calculation of α_λ thus reduces to the calculation of characters of the form $\chi_\lambda^{m/d}$.

It is the purpose of this note to suggest, in §2, a method of calculation of these characters, and, in §3, to indicate some relations satisfied by the α_λ .

2. The well known formulae for $\chi_{1^m}^\lambda$ are

$$\chi_{1^m}^\lambda = \frac{m! \prod_{r < s} (\lambda_r - \lambda_s - r + s)}{\prod_r (\lambda_r + p - r)!}$$

where p = number of parts in (λ) ;

$$\chi_{1^m}^\lambda = \frac{m!}{\prod_{r,s} (\lambda_r + \bar{\lambda}_s - r - s + 1)},$$

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where $(\bar{\lambda}) = (\bar{\lambda}_1, \bar{\lambda}_2, \dots)$ is the partition conjugate to (λ) ;

$$\chi_{1^m}^\lambda = \frac{m!}{H_\lambda},$$

where H_λ is the product of the hook-lengths h_{ij} in the hook-graph of (λ) [3]. These give the coefficient of $\{\lambda\}$ in s_1^m . An alternative method, which extends to $\chi_r^{\lambda m/r}$, the coefficient of $\{\lambda\}$ in $s_r^{m/r}$, is by evaluation of certain determinants; [4, pp. 134–135; 5]. Thus to find $\chi_{10}^{531^2}$ we evaluate

$$10! \begin{vmatrix} 1 & 1 & 1 & 1 \\ 5! & 6! & 7! & 8! \\ 1 & 1 & 1 & 1 \\ 2! & 3! & 4! & 5! \\ \cdot & 1 & 1 & 1 \\ & & 1! & 2! \\ \cdot & \cdot & 1 & 1 \\ & & & 1! \end{vmatrix}, \text{ or preferably } 10! \begin{vmatrix} 35 & 4 \\ 8! & 5! \\ 1 & 1 \\ 5! & 2! \end{vmatrix},$$

obtained respectively from

$$\{531^2\} = \begin{vmatrix} \{5\} & \{6\} & \{7\} & \{8\} \\ \{2\} & \{3\} & \{4\} & \{5\} \\ \cdot & \{0\} & \{1\} & \{2\} \\ \cdot & \cdot & \{0\} & \{1\} \end{vmatrix} \text{ and } \{531^2\} = \begin{vmatrix} \{51^3\} & \{21^3\} \\ \{5\} & \{2\} \end{vmatrix},$$

[6] giving 567. To find the coefficient of $\{531^2\}$ in $s_r^{10/r}$ we replace every $\{k\}$ in the first of these determinants by zero if k is not a multiple of r , and by $1/(k/r)!$ if it is. Multiplying the determinant by $(10/r)!$ gives the required coefficient. Thus the coefficients of $\{531^2\}$ in s_2^5, s_5^2, s_{10} are respectively

$$5! \begin{vmatrix} \cdot & \frac{1}{3!} & \cdot & \frac{1}{4!} \\ 1 & \cdot & \frac{1}{2!} & \cdot \\ \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{vmatrix} = 15, \quad 2! \begin{vmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{vmatrix} = 2, \text{ and zero,}$$

giving the coefficient of $\{531^2\}$ in γ_{10} as 55. This was given as 53 by Thrall [1], but was later corrected by Brandt [2].

Justification of the above procedure is given conveniently by the use of the differential operator D_λ obtained from $\{\lambda\}$ by replacing s_i^a by $i^a \partial^a / \partial s_i^a$, [7]. Thus the coefficient of $\{\lambda\}$ in s_1^m is $D_\lambda s_1^m$, and if $(\lambda) = (\lambda_1, \lambda_2, \dots, \lambda_p)$, the coefficient can be written as

$$\begin{vmatrix} D_{\lambda_1} & D_{\lambda_1+1} & \cdots & D_{\lambda_1+p-1} \\ D_{\lambda_2-1} & D_{\lambda_2} & \cdots & D_{\lambda_2+p-2} \\ D_{\lambda_3-2} & D_{\lambda_3-1} & \cdots & D_{\lambda_3+p-3} \\ \cdots & \cdots & \cdots & \cdots \\ D_{\lambda_{p-p+1}} & \cdots & \cdots & D_{\lambda_p} \end{vmatrix} s_1^m$$

and the only effective part of each D_k is $(k!)^{-1} \partial^k / \partial s_1^k$. Then in $D_\lambda s_1^{m/r}$ the only effective part of each D_k is

$$\frac{1}{k!} \frac{k!}{r^{k/r} (k/r)!} r^{k/r} \frac{\partial^{k/r}}{\partial s_1^{k/r}} = \frac{1}{(k/r)!} \frac{\partial^{k/r}}{\partial s_1^{k/r}}$$

so that we get zero elements in the determinant when k is not a multiple of r , and elements $1/(k/r)!$ when k is a multiple of r .

3. We now list a number of results concerning the α_λ .

I. When m is prime, α_λ is the integer nearest to $m^{-1} \chi_1^\lambda$ [2].

II. $\alpha_m = 0$ for $m > 1$, [10], and $\alpha_1^m = 0$ for $m > 2$.

These follow since $\alpha_m = m^{-1} \sum \mu(d)$, and $\alpha_1^m = m^{-1} \sum (-1)^{m+m/d} \mu(d)$.

III. $\alpha_{m-1,1} = 1$ for $m > 1$, [10], and $\alpha_{21}^{m-2} = 1$ for $m > 2$.

These follow since

$$\chi_1^{m-1,1} = m - 1, \quad \text{and} \quad \chi_a^{m-1,1} = -1 \quad \text{for } a > 1, b \geq 1,$$

[8, p. 137] and so

$$\alpha_{m-1,1} = m^{-1} \left[m - 1 - \sum_{d \neq 1} \mu(d) \right] = 1 \quad \text{for } m > 1,$$

$$\alpha_{21}^{m-2} = m^{-1} \left[m - 1 - \sum_{d \neq 1} (-1)^{m+m/d} \mu(d) \right] = 1, \quad \text{for } m > 2.$$

IV. If m is odd or a multiple of four, then $\alpha_\lambda = \alpha_{\tilde{\lambda}}$. It is well known that $\chi_{a^{m/d}}^\lambda = \chi_{a^{m/d}}^{\tilde{\lambda}}$ for odd values of d , and also for even d whenever m/d is even. This proves IV.

When m is twice an odd integer, $\chi_{a^{m/d}}^\lambda = -\chi_{a^{m/d}}^{\tilde{\lambda}}$ for even d , and in this case γ_m , expressed in power sums, can be written as $P + Q$, where the coefficients of $\{\lambda\}$ and $\{\tilde{\lambda}\}$ are the same in P , but have opposite signs in Q . P has all $s_d^{m/d}$ with odd d , and Q has all $s_d^{m/d}$ with even d . There are certain partitions (λ) for which the coefficient of $\{\lambda\}$ in Q is zero, and for these $\alpha_\lambda = \alpha_{\tilde{\lambda}}$.

The following three results are typical of many which can be obtained by equating appropriate coefficients when the right hand side of

$$m^{-1} \sum_{d|m} \mu(d) s_d^{m/d} = \sum_{\lambda} \alpha_{\lambda} \{ \lambda \}$$

is expressed in terms of power sums by writing

$$\{ \lambda \} = (m!)^{-1} \sum_{\rho} h_{\rho} \chi_{\rho}^{\lambda} S_{\rho},$$

where h_{ρ} is the order of the class $(\rho) = 1^{x_1} 2^{x_2} \dots$ of \mathfrak{S}_m , and $S_{\rho} = s_1^{x_1} s_2^{x_2} \dots$.

V. $\sum_{\lambda} \chi_{1^m}^{\lambda} \alpha_{\lambda} = (m-1)!$ This is obtained by equating coefficients of s_1^m .

VI. $\sum_{\lambda=(m-r, 1^r)} (-1)^r \alpha_{\lambda} = \mu(m)$. Obtained by equating coefficients of s_m .

VII. If S -functions of ranks one and two are written respectively in Frobenius notation as

$$\{ \lambda \} = \left\{ \begin{matrix} X_r \\ Y_r \end{matrix} \right\} \quad \text{and} \quad \{ \nu \} = \left\{ \begin{matrix} X_{t_1} & X_{t_2} \\ Y_{t_1} & Y_{t_2} \end{matrix} \right\},$$

and a, b are any two unequal positive integers such that $a+b=m$, then $\sum_{\lambda} \theta_r \alpha_{\lambda} + \sum_{\nu} \phi_t \alpha_{\nu} = 0$, where

$$\begin{aligned} \theta_r &= (-1)^{Y_r} && \text{if } m > X_r \geq a, \\ &= 0 && \text{if } a > X_r \geq b, \\ &= (-1)^{Y_r+1} && \text{if } b > X_r \geq 0, \\ \phi_t &= (-1)^{Y_{t_1}+Y_{t_2}} && \text{if } X_{t_1} + Y_{t_1} = a - 1, \\ &= (-1)^{Y_{t_1}+Y_{t_2}+1} && \text{if } X_{t_1} + Y_{t_2} = a - 1, \\ &&& \text{or if } X_{t_2} + Y_{t_1} = a - 1, \\ &= 0 && \text{otherwise.} \end{aligned}$$

This result follows by equating coefficients of $s_a s_b$, [9]. Other special relations of this kind may be obtained by equating coefficients of $s_a s_b s_c$, and of other terms. They depend on a knowledge of expressions such as θ_r, ϕ_t above for the appropriate characteristics of \mathfrak{S}_m .

VIII. If h_d is the order of the class $(d^{m/d})$ of \mathfrak{S}_m , then

$$\sum_{\lambda} \alpha_{\lambda}^2 = \frac{(m-1)!}{m} \sum_d \frac{1}{h_d},$$

for square free values of d .

We have $D_{\gamma_m} \gamma_m = \sum_{\lambda} \alpha_{\lambda} D_{\lambda} \sum_{\lambda} \alpha_{\lambda} \{ \lambda \} = \sum_{\lambda} \alpha_{\lambda}^2$. Also

$$\begin{aligned}
 D_{\gamma_m} \gamma_m &= m^{-1} \left[\sum_d \mu(d) d^{m/d} \frac{\partial^{m/d}}{\partial s_d^{m/d}} \right] m^{-1} \sum_d \mu(d) s_d^{m/d} \\
 &= (m^2)^{-1} \left[\sum (\mu(d))^2 d^{m/d} (m/d)! \right] \\
 &= (m^2)^{-1} \sum \frac{m!}{h_d}
 \end{aligned}$$

for square free values of d .

More explicit results may be written down when m has some prescribed form. Thus when m is a prime p , $\sum \alpha_\lambda^2 = p^{-1} [1 + (p-1)!]$, and when m is the product of two distinct primes p, q ,

$$\sum \alpha_\lambda^2 = \frac{1}{pq} [(pq-1)! + p^{q-1}(q-1)! + q^{p-1}(p-1)! + 1].$$

When m is not twice an odd number we have $\alpha_\lambda = \alpha_{\bar{\lambda}}$ by IV. But if $m = 2(2k+1)$, then by evaluating $D_{\gamma_m} \tilde{\gamma}_m$ in two ways, we obtain a further result;

IX. If m is twice an odd number, then $\sum_\lambda \alpha_\lambda \alpha_{\bar{\lambda}} = (m-1)!/m \cdot \sum_d (-1)^{d+1} (h_d)^{-1}$ for square free values of d .

Some results on the characters of the Lie representations in the special case when the number of generators is two have been given in a recent paper by Davis [10].

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