PROXIMITY MAPS FOR CONVEX SETS

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The method of successive approximation is applied to the problem of obtaining points of minimum distance on two convex sets. Specifically, given a closed convex set $K$ in Hilbert space, let $P$ be the map which associates with each point $x$ the point $Px$ of $K$ closest to $x$. That $P$ is well-defined is proved in [1, p. 6]. $P$ will be called the proximity map for $K$. If there are two such sets, $K_1$ and $K_2$, let $Q$ denote the composition $P_1P_2$ of their proximity maps. It is shown that every fixed point of $Q$ is a point of $K_1$ closest to $K_2$, and that the fixed points of $Q$ may be obtained by iteration of $Q$ when one of the sets is compact or when both are polytopes in $E^m$. An application to the solution of linear inequalities is cited. Our thanks are due the referee for having suggested substantial simplifications.

Theorem 1. Let $Q$ be a map of a metric space into itself such that

(i) $d(Qx, Qy) \leq d(x, y)$,

(ii) if $x \neq Qx$, then $d(Qx, Q^2x) < d(x, Qx)$,

(iii) for each $x$, the sequence $Q^n x$ has a cluster point. Then for each $x$, the sequence $Q^n x$ converges to a fixed point of $Q$.

Proof. By (i), the sequence $d(Q^n x, Q^{n+1} x)$ is nonincreasing. Let $y$ be a cluster point of $Q^n x$, say $y = \lim_k Q^k x$. By (i), $Q$ is continuous; therefore $d(y, Qy) = \lim_k d(Q^k x, Q^{k+1} x) = \lim_n d(Q^n x, Q^{n+1} x) = \lim_k d(Q^{n+k+1}, Q^{n+k+2} x) = d(Qy, Q^2 y)$ contrary to (ii) unless $y = Qy$. From (i) it follows that for all $n$, $d(Q^{N+n} x, y) \leq d(Q^N x, y)$ whence $Q^n x \rightarrow y$.

Corollary. Let $Q$ be a map of a normed linear space into itself having the property $\|Qx - Qy\| \leq \|x - y\|$, equality holding only if $x = y$. Let $R = \alpha Q + (1 - \alpha) I$, $(0 < \alpha \leq 1)$. If the range of $R$ is compact, then $Q$ has a unique fixed point which is the limit of every sequence $R^n x$ with $x$ arbitrary. (For related results, see [2].)

Lemma. Let $K$ be a convex set in Hilbert space. A point $b \in K$ is nearest a point $a \in K$ if and only if $s = (x - \overline{a}) b - (b \overline{a}) \geq 0$ for all $x \in K$.

Proof. Suppose $b$ nearest $a$, and let $x$ be arbitrary in $K$. When $0 \leq t \leq 1$, $tx + (1 - t) b \in K$. Thus $0 \leq \|a - tx - (1 - t) b\|^2 - \|a - b\|^2 = t^2 \|b - x\|^2 + 2ts$. But this inequality would be violated by small $t$.
unless \( s \geq 0 \). For the converse, suppose \( s \geq 0 \). Then 
\[
\|x - a\|^2 - \|a - b\|^2 = (x, x) - 2(a, x) + 2(a, b) - (b, b) = (x - b, x - b) + 2s \geq 0.
\]

**Theorem 2.** Let \( K_1 \) and \( K_2 \) be two closed convex sets in Hilbert space. Let \( P_i \) denote the proximity map for \( K_i \). Any fixed point of \( P_1 P_2 \) is a point of \( K_1 \) nearest \( K_2 \), and conversely.

**Proof.** Suppose \( y = P_2 x \) and \( x = P_1 y \). If \( x = y \), the distance between \( K_1 \) and \( K_2 \) is thereby attained. Otherwise \( x \notin K_2 \) and \( y \notin K_1 \). If \( u \) is arbitrary in \( K_1 \), then by the Lemma, \((u - x, x - y) \geq 0\) whence \((u, x - y) \geq (x, x - y)\). Similarly for arbitrary \( v \in K_2 \), \((v, y - x) \geq (y, y - x)\). Addition yields \((u - v, x - y) \geq (x - y, x - y)\) from which via the Schwartz inequality, \(\|u - v\| \geq \|x - y\|\). For the converse, suppose that \(\|x - P_2 x\| \leq \|z - P_2 z\|\) for all \( z \in K_1 \). Setting \( z = P_1 P_2 x \) we have \(\|z - P_2 z\| \leq \|x - P_2 x\| \leq \|z - P_2 z\|\) whence \(z = x\) by the uniqueness of \(z\).

**Theorem 3.** The proximity map \( P \) for a closed convex set \( K \) in Hilbert space satisfies the Lipschitz condition \(\|Px - Py\| \leq \|x - y\|\), equality holding only if \(\|x - Px\| = \|y - Py\|\).

**Proof.** By the Lemma, \( A \equiv (Px - Py, Py - y) \geq 0 \) and \( B \equiv (Py - Px, Px - x) \geq 0 \). Regrouping terms in the inequality \(A + B \geq 0\) and using the Schwartz inequality, one has 
\[
\|Py - Px\| \cdot \|y - x\| \geq \langle Py - Px, y - x \rangle \geq \langle Py - Px, Py - Px \rangle = \|Py - Px\|^2,
\]
whence \(\|y - x\| \geq \|Py - Px\|\). Equality holds here only if \(A = B = 0\) and if \(Py - Px = \lambda(y - x)\). Thus \(C \equiv (y - x, Py - y) = 0\) and \(D \equiv (y - x, Px - x) = 0\). A computation shows then that \(0 = A - B + C + D = \|Px - x\|^2 - \|Py - y\|^2\).

**Theorem 4.** Let \( K_1 \) and \( K_2 \) be two closed convex sets in Hilbert space and \( Q \) the composition \( P_1 P_2 \) of their proximity maps. Convergence of \( Q^n x \) to a fixed point of \( Q \) is assured when either (a) one set is compact, or (b) one set is finite dimensional and the distance between the sets is attained.

**Proof.** Theorem 3 implies that \( Q \) satisfies (i) of Theorem 1. If \( y = Qx \neq x \in K_1 \), then \(\|y - P_2 y\| \leq \|y - P_2 x\| \leq \|x - P_2 x\|\). By Theorem 3, \(\|Qx - Qy\| \leq \|P_2 x - P_2 y\| \leq \|x - y\|\). Thus \( Q \) satisfies (ii) of Theorem 1. If the distance between the sets is attained, then \( Q \) has a fixed point \( y \) by Theorem 2, and \(\|Q^n x\| \leq \|Q^n x - Qy\| + \|y\| \leq \|x - y\| + \|y\|\). Boundedness of \(\|Q^n x\|\) and finite-dimensionality of \( K_1 \) suffice for (iii) of Theorem 1. \( Q' = P_2 P_1 \) replaces \( Q \) in these arguments when \( K_2 \) replaces \( K_1 \). But if \( y \) is a fixed point of \( Q' \), then \( P_1 y \) is a fixed point of \( Q \).

**Theorem 5.** In a finite dimensional Euclidean space, the distance
between two polytopes is attained, a polytope being the intersection of a finite family of halfspaces.

Proof. First, the case when both polytopes are linear manifolds. Let \( K_1 \) be the linear span of \( \{x_1, \ldots, x_m\} \) and \( K_2 \) the linear span of \( \{y_1, \ldots, y_n\} \) translated by a vector \( cy_0 \). Assume the \( x \)'s and \( y \)'s each form orthonormal sets. A point \( x = \sum \xi_i x_i \) is sought which minimizes \( G = \|cy_0 + \sum (x, y_i)y_i - x\|^2 \). \( G \) is a positive definite quadratic function of \( \xi, \ldots, \xi_m \), and therefore attains a minimum.

Now assume the validity of the theorem when one polytope is of dimension less than \( n \) and the other is a linear manifold. (The validity for \( n=1 \) being established by the above.) Let \( K_1 \) be a polytope of dimension \( n \) and \( K_2 \) a linear manifold. On each proper face of \( K_1 \) there is a point nearest \( K_2 \). There being only a finite number of faces, either one of these points is the required one or there is a point \( x_0 \in K_1 \) such that \( d(x_0, K_2) < d(F, K_2) \) for all proper faces \( F \) of \( K_1 \). In the latter case, let \( y_0 \) be chosen nearest to \( K_2 \) in the least linear manifold containing \( K_1 \). Since \( d(x, K_2) \) is a convex function of \( x \), the line segment \( x_0y_0 \) contains no point of any proper face of \( K_1 \). Therefore \( y_0 \in K_1 \) and must be the required point. The proof for the remaining case is as above, mutatis mutandis.

Linear inequalities. Consider the system of linear inequalities \((A^i, x) \leq b_i\) where \( x \in E_n \) and \( 1 \leq i \leq m \). Let \( K_1 \) denote the range of the matrix \( A \) and \( K_2 \) the orthant \( \{y \in E_m: y_i \leq b_i \text{ all } i\} \). If \( K_1 \) and \( K_2 \) have a point in common, then any \( x \) for which \( Ax \in K_1 \cap K_2 \) is a solution of the system. Even if the system is inconsistent, a point on \( K_1 \) closest to \( K_2 \) may be obtained by iteration of the map \( Q = P_1 P_2 \). These proximity maps are defined as follows. If \( A \) is of rank \( n \), \( P_1 = A(A^T A)^{-1} A^T \). If \( A \) is not necessarily of rank \( n \), one may write \( P_1 = BB^T \) where \( B \) is a column-orthogonal matrix whose range is that of \( A \). \( P_2 y = z \) if and only if \( z_i = y_i \) when \( y_i \leq b_i \) and \( z_i = b_i \) otherwise. Convergence of each sequence \( Q^i x \) is guaranteed by Theorem 4.

Bibliography


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