PROXIMITY MAPS FOR CONVEX SETS

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The method of successive approximation is applied to the problem of obtaining points of minimum distance on two convex sets. Specifically, given a closed convex set K in Hilbert space, let P be the map which associates with each point x the point Px of K closest to x. That P is well-defined is proved in [1, p. 6]. P will be called the *proximity map* for K. If there are two such sets, K_1 and K_2, let Q denote the composition P_1P_2 of their proximity maps. It is shown that every fixed point of Q is a point of K_1 closest to K_2, and that the fixed points of Q may be obtained by iteration of Q when one of the sets is compact or when both are polytopes in E_n. An application to the solution of linear inequalities is cited. Our thanks are due the referee for having suggested substantial simplifications.

THEOREM 1. *Let Q be a map of a metric space into itself such that*
(i) $d(Qx, Qy) \leqq d(x, y)$,
(ii) *if* $x \neq Qx$, *then* $d(Qx, Q^2x) < d(x, Qx)$,
(iii) *for each x, the sequence Q^nx has a cluster point. Then for each x, the sequence Q^nx converges to a fixed point of Q.*

PROOF. By (i), the sequence $d(Q^nx, Q^{n+1}x)$ is nonincreasing. Let y be a cluster point of Q^nx, say $y = \lim_k Q^{n_k}x$. By (i), Q is continuous; therefore $d(y, Qy) = \lim_k d(Q^{n_k}x, Q^{n_k+1}x) = \lim_n d(Q^nx, Q^{n+1}x) = \lim_k d(Q^{n_k+1}, Q^{n_k+2}x) = d(Qy, Q^2y)$ contrary to (ii) unless $y = Qy$. From (i) it follows that for all n, $d(Q^{N+n}x, y) \leqq d(Q^Nx, y)$ whence $Q^nx \to y$.

COROLLARY. *Let Q be a map of a normed linear space into itself having the property $\|Qx - Qy\| \leqq \|x - y\|$, equality holding only if $x = y$. Let $R = \alpha Q + (1-\alpha)I$, $(0 < \alpha \leqq 1)$. If the range of R is compact, then Q has a unique fixed point which is the limit of every sequence R^nx with x arbitrary. (For related results, see [2].)*

LEMMA. *Let K be a convex set in Hilbert space. A point $b \in K$ is nearest a point $a \notin K$ if and only if $s \equiv (x - b, b - a) \geqq 0$ for all $x \in K$.*

PROOF. Suppose b nearest a, and let x be arbitrary in K. When $0 \leqq t \leqq 1$, $tx + (1-t)b \in K$. Thus $0 \leqq \|a - tx - (1-t)b\|^2 - \|a - b\|^2 = t^2\|b - x\|^2 + 2ts$. But this inequality would be violated by small t

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unless $s \geqq 0$. For the converse, suppose $s \geqq 0$. Then $\|x-a\|^2 - \|a-b\|^2 = (x, x) - 2(a, x) + 2(a, b) - (b, b) = (x-b, x-b) + 2s \geqq 0$.

THEOREM 2. *Let K_1 and K_2 be two closed convex sets in Hilbert space. Let P_i denote the proximity map for K_i. Any fixed point of P_1P_2 is a point of K_1 nearest K_2, and conversely.*

PROOF. Suppose $y = P_2x$ and $x = P_1y$. If $x = y$, the distance between K_1 and K_2 is thereby attained. Otherwise $x \notin K_2$ and $y \notin K_1$. If u is arbitrary in K_1, then by the Lemma, $(u-x, x-y) \geqq 0$ whence $(u, x-y) \geqq (x, x-y)$. Similarly for arbitrary $v \in K_2$, $(v, y-x) \geqq (y, y-x)$. Addition yields $(u-v, x-y) \geqq (x-y, x-y)$ from which via the Schwartz inequality, $\|u-v\| \geqq \|x-y\|$. For the converse, suppose that $\|x-P_2x\| \leqq \|z-P_2z\|$ for all $z \in K_1$. Setting $z = P_1P_2x$ we have $\|z-P_2z\| \leqq \|z-P_2x\| \leqq \|x-P_2x\| \leqq \|z-P_2z\|$ whence $z = x$ by the uniqueness of z.

THEOREM 3. *The proximity map P for a closed convex set K in Hilbert space satisfies the Lipschitz condition $\|Px-Py\| \leqq \|x-y\|$, equality holding only if $\|x-Px\| = \|y-Py\|$.*

PROOF. By the Lemma, $A \equiv (Px-Py, Py-y) \geqq 0$ and $B \equiv (Py-Px, Px-x) \geqq 0$. Regrouping terms in the inequality $A+B \geqq 0$ and using the Schwartz inequality, one has $\|Py-Px\| \cdot \|y-x\| \geqq (Py-Px, y-x) \geqq (Py-Px, Py-Px) = \|Py-Px\|^2$, whence $\|y-x\| \geqq \|Py-Px\|$. Equality holds here only if $A = B = 0$ and if $Py - Px = \lambda(y-x)$. Thus $C \equiv (y-x, Py-y) = 0$ and $D \equiv (y-x, Px-x) = 0$. A computation shows then that $0 = A - B + C + D = \|Px-x\|^2 - \|Py-y\|^2$.

THEOREM 4. *Let K_1 and K_2 be two closed convex sets in Hilbert space and Q the composition P_1P_2 of their proximity maps. Convergence of Q^nx to a fixed point of Q is assured when either* (a) *one set is compact, or* (b) *one set is finite dimensional and the distance between the sets is attained.*

PROOF. Theorem 3 implies that Q satisfies (i) of Theorem 1. If $y \equiv Qx \neq x \in K_1$, then $\|y-P_2y\| \leqq \|y-P_2x\| < \|x-P_2x\|$. By Theorem 3, $\|Qx-Qy\| \leqq \|P_2x-P_2y\| < \|x-y\|$. Thus Q satisfies (ii) of Theorem 1. If the distance between the sets is attained, then Q has a fixed point y by Theorem 2, and $\|Q^nx\| \leqq \|Q^nx-Qy\| + \|y\| \leqq \|x-y\| + \|y\|$. Boundedness of $\|Q^nx\|$ and finite-dimensionality of K_1 suffice for (iii) of Theorem 1. $Q' = P_2P_1$ replaces Q in these arguments when K_2 replaces K_1. But if y is a fixed point of Q', then P_1y is a fixed point of Q.

THEOREM 5. *In a finite dimensional Euclidean space, the distance* *between two polytopes is attained, a polytope being the intersection of a finite family of halfspaces.*

PROOF. First, the case when both polytopes are linear manifolds. Let K_1 be the linear span of $\{x_1, \cdots, x_m\}$ and K_2 the linear span of $\{y_1, \cdots, y_n\}$ translated by a vector cy_0. Assume the x's and y's each form orthonormal sets. A point $x=\sum \xi_i x_i$ is sought which minimizes $G=\|cy_0+\sum(x, y_i)y_i-x\|^2$. G is a positive definite quadratic function of $\xi_1, \cdots, \xi_m$, and therefore attains a minimum.

Now assume the validity of the theorem when one polytope is of dimension less than n and the other is a linear manifold. (The validity for $n=1$ being established by the above.) Let K_1 be a polytope of dimension n and K_2 a linear manifold. On each proper face of K_1 there is a point nearest K_2. There being only a finite number of faces, either one of these points is the required one or there is a point $x_0 \in K_1$ such that $d(x_0, K_2) < d(F, K_2)$ for all proper faces F of K_1. In the latter case, let y_0 be chosen nearest to K_2 in the least linear manifold containing K_1. Since $d(x, K_2)$ is a convex function of x, the line segment x_0y_0 contains no point of any proper face of K_1. Therefore $y_0 \in K_1$ and must be the required point. The proof for the remaining case is as above, mutatis mutandis.

LINEAR INEQUALITIES. Consider the system of linear inequalities $(A^i, x) \leqq b_i$ where $x \in E_n$ and $1 \leqq i \leqq m$. Let K_1 denote the range of the matrix A and K_2 the orthant $\{y \in E_m : y_i \leqq b_i \text{ all } i\}$. If K_1 and K_2 have a point in common, then any x for which $Ax \in K_1 \cap K_2$ is a solution of the system. Even if the system is inconsistent, a point on K_1 closest to K_2 may be obtained by iteration of the map $Q=P_1P_2$. These proximity maps are defined as follows. If A is of rank n, $P_1=A(A^TA)^{-1}A^T$. If A is not necessarily of rank n, one may write $P_1=BB^T$ where B is a column-orthogonal matrix whose range is that of A. $P_2y=z$ if and only if $z_i=y_i$ when $y_i \leqq b_i$ and $z_i=b_i$ otherwise. Convergence of each sequence Q^nx is guaranteed by Theorem 4.

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