EACH HOMOGENEOUS NONDEGENERATE CHAINABLE CONTINUUM IS A PSEUDO-ARC

R. H. BING

The endeavor to find all homogeneous plane continua continues. The simple closed curve and the point are obvious examples. The discovery of the pseudo-arc [1; 6] should have exploded (but did not) the conjectures that there are no others. A history of the problem with a discussion of various false starts is given in [4]. Finding the circle of pseudo-arcs [4] raised the number of known examples to four. Are there others as yet undiscovered? Jones showed [5] that each one which does not separate the plane is indecomposable. The theorem in this paper narrows the field for search still further.

We recall the following definitions:
A set $X$ is homogeneous if for each pair of points $p, q$ of $X$ there is a homeomorphism of $X$ onto itself that takes $p$ onto $q$.
A continuum is nondegenerate if it contains more than one point.
An $\epsilon$-chain is a finite ordered collection $d_1, d_2, \ldots, d_n$ of open sets, each of diameter less than $\epsilon$, such that $d_i$ intersects $d_j$ if and only if $i$ and $j$ are adjacent integers.

A snakelike or chainable continuum is a compact metric continuum $M$ such that for each positive number $\epsilon$, $M$ can be covered by an $\epsilon$-chain.
A point $p$ is an end point of a snakelike continuum $M$ if for each positive number $\epsilon$ there is an $\epsilon$-chain covering $M$ such that the first link of the chain contains $p$.
A continuum is indecomposable if it is not the sum of two proper subcontinua. It is hereditarily indecomposable if each subcontinuum of it is indecomposable.
A pseudo-arc is a nondegenerate, hereditarily indecomposable, chainable continuum. Any two such continua are homeomorphic [2].

Theorem. Each homogeneous, nondegenerate, chainable continuum is a pseudo-arc.

Proof. First we show that $M$ has an end point $p$. For each integer $n$, let $q_n$ be a point of $M$ such that a $1/n$-chain covers $M$ and an end link of this chain contains $q_n$. Some subsequence of $q_1, q_2, \ldots$ converges to a point $q$. Then the point $q$ of $M$ has the following property:

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Property of \(q\). For each neighborhood \(N\) of \(q\) and each positive number \(\epsilon\) there is an \(\epsilon\)-chain covering \(M\) one of whose end links intersects \(M\) and lies in \(N\). It follows from the homogeneity of \(M\) and the fact that each homeomorphism of \(M\) onto itself is uniformly continuous that each point of \(M\) has the Property of \(q\).

Let \(d_1\) be an end link of a 1-chain covering \(M\) such that \(d_1\) contains a point \(p_1\) of \(M\). Since \(p_1\) has the Property of \(q\), there is an end link \(d_2\) of a 1/2-chain covering \(M\) such that \(d_1\) contains \(d_2\) and \(d_2\) contains a point \(p_2\) of \(M\). Also, there is an end link \(d_3\) of a 1/3-chain covering \(M\) such that \(d_2\) contains \(d_3\) and \(d_3\) contains a point \(p_3\) of \(M\). Similarly, we obtain \(d_4, d_5, \ldots\). Then the point \(p\) which is the intersection of \(d_1, d_2, \ldots\) is an end point of \(M\).

Finally we show that \(M\) is hereditarily indecomposable. Assume \(M\) contains a continuum \(H\) which is the sum of two proper subcontinua \(H', H''\). Let \(p\) be a point of \(H' \cdot H''\). Then it follows from the homogeneity of \(M\) that \(p\) is an end point of \(M\). However, as noted in [3], this would imply that one of \(H', H''\) contains the other and this is impossible since each is a proper subcontinuum of their sum.

References


University of Wisconsin