

# EACH HOMOGENEOUS NONDEGENERATE CHAINABLE CONTINUUM IS A PSEUDO-ARC

R. H. BING<sup>1</sup>

The endeavor to find all homogeneous plane continua continues. The simple closed curve and the point are obvious examples. The discovery of the pseudo-arc [1; 6] should have exploded (but did not) the conjectures that there are no others. A history of the problem with a discussion of various false starts is given in [4]. Finding the circle of pseudo-arcs [4] raised the number of known examples to four. Are there others as yet undiscovered? Jones showed [5] that each one which does not separate the plane is indecomposable. The theorem in this paper narrows the field for search still further.

We recall the following definitions:

A set  $X$  is *homogeneous* if for each pair of points  $p, q$  of  $X$  there is a homeomorphism of  $X$  onto itself that takes  $p$  onto  $q$ .

A continuum is *nondegenerate* if it contains more than one point.

An  $\epsilon$ -*chain* is a finite ordered collection  $d_1, d_2, \dots, d_n$  of open sets, each of diameter less than  $\epsilon$ , such that  $d_i$  intersects  $d_j$  if and only if  $i$  and  $j$  are adjacent integers.

A *snakelike* or *chainable* continuum is a compact metric continuum  $M$  such that for each positive number  $\epsilon$ ,  $M$  can be covered by an  $\epsilon$ -chain.

A point  $p$  is an *end point* of a snakelike continuum  $M$  if for each positive number  $\epsilon$  there is an  $\epsilon$ -chain covering  $M$  such that the first link of the chain contains  $p$ .

A continuum is *indecomposable* if it is not the sum of two proper subcontinua. It is *hereditarily indecomposable* if each subcontinuum of it is indecomposable.

A *pseudo-arc* is a nondegenerate, hereditarily indecomposable, chainable continuum. Any two such continua are homeomorphic [2].

**THEOREM.** *Each homogeneous, nondegenerate, chainable continuum is a pseudo-arc.*

**PROOF.** First we show that  $M$  has an end point  $p$ . For each integer  $n$ , let  $q_n$  be a point of  $M$  such that a  $1/n$ -chain covers  $M$  and an end link of this chain contains  $q_n$ . Some subsequence of  $q_1, q_2, \dots$  converges to a point  $q$ . Then the point  $q$  of  $M$  has the following property:

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Presented to the Society, February 23, 1957; received by the editors October 16, 1958.

Work on this paper was supported by Contract NSF G1219.

*Property of  $q$ .* For each neighborhood  $N$  of  $q$  and each positive number  $\epsilon$  there is an  $\epsilon$ -chain covering  $M$  one of whose end links intersects  $M$  and lies in  $N$ . It follows from the homogeneity of  $M$  and the fact that each homeomorphism of  $M$  onto itself is uniformly continuous that each point of  $M$  has the *Property of  $q$* .

Let  $d_1$  be an end link of a 1-chain covering  $M$  such that  $d_1$  contains a point  $p_1$  of  $M$ . Since  $p_1$  has the *Property of  $q$* , there is an end link  $d_2$  of a  $1/2$ -chain covering  $M$  such that  $d_1$  contains  $\bar{d}_2$  and  $d_2$  contains a point  $p_2$  of  $M$ . Also, there is an end link  $d_3$  of a  $1/3$ -chain covering  $M$  such that  $d_2$  contains  $\bar{d}_3$  and  $d_3$  contains a point  $p_3$  of  $M$ . Similarly, we obtain  $d_4, d_5, \dots$ . Then the point  $p$  which is the intersection of  $d_1, d_2, \dots$  is an end point of  $M$ .

Finally we show that  $M$  is hereditarily indecomposable. Assume  $M$  contains a continuum  $H$  which is the sum of two proper subcontinua  $H', H''$ . Let  $p$  be a point of  $H' \cdot H''$ . Then it follows from the homogeneity of  $M$  that  $p$  is an end point of  $M$ . However, as noted in [3], this would imply that one of  $H', H''$  contains the other and this is impossible since each is a proper subcontinuum of their sum.

#### REFERENCES

1. R. H. Bing, *A homogeneous indecomposable plane continuum*, Duke Math. J. vol. 15 (1948) pp. 729-742.
2. ———, *Concerning hereditarily indecomposable continua*, Pacific J. Math. vol. 1 (1951) pp. 43-51.
3. ———, *Snake-like continua*, Duke Math. J. vol. 18 (1951) pp. 653-663.
4. R. H. Bing and F. B. Jones, *Another homogeneous plane continuum*, Trans. Amer. Math. Soc. vol. 90 (1959) pp. 171-192.
5. F. B. Jones, *Certain homogeneous unicoherent indecomposable continua*, Proc. Amer. Math. Soc. vol. 2 (1951) pp. 855-859.
6. E. E. Moise, *A note on the pseudo-arc*, Trans. Amer. Math. Soc. vol. 64 (1949) pp. 57-58.

UNIVERSITY OF WISCONSIN