

# ON PHI-FAMILIES

CHARLES E. WATTS

1. **Introduction.** The purpose of this note is to show that the notion of sections with support in a phi-family in the Cartan version of the Leray theory of sheaves can be avoided by the following expedient. One uses the phi-family to construct a new space in the manner of a one-point compactification. Then a sheaf on the original space is shown to yield a new sheaf on the new space whose cohomology (with unrestricted supports) is that of the original sheaf with restricted supports. A partial generalization to the Grothendieck cohomology theory is given.

2. **Phi-families.** A family  $\mathcal{F}$  of subsets of a topological space  $X$  is a *family of supports* [1] provided:

- (I) each member of  $\mathcal{F}$  is closed;
- (II) if  $F \in \mathcal{F}$ , then each closed subset of  $F$  is  $\in \mathcal{F}$ ;
- (III) if  $F_1, F_2 \in \mathcal{F}$ , then  $F_1 \cup F_2 \in \mathcal{F}$ .

Given a family of supports  $\mathcal{F}$ , we choose an object  $\infty \in X$ , and define  $X' = \cup \mathcal{F}$ ,  $X^* = X' \cup \{\infty\}$ . We then topologize  $X^*$  by saying that a subset  $U$  of  $X^*$  is open iff either  $U$  is open in  $X'$  or else  $X^* - U \in \mathcal{F}$ . It is readily verified that these open sets in fact form a topology for  $X^*$  and that the inclusion  $X' \subset X^*$  is a topological imbedding.

The family of supports  $\mathcal{F}$  is a *phi-family* [3] provided also:

- (IV) each member of  $\mathcal{F}$  has a closed neighborhood in  $\mathcal{F}$ ;
- (V) each member of  $\mathcal{F}$  is paracompact.

**PROPOSITION 1.** *If  $\mathcal{F}$  is a phi-family, then  $X^*$  is paracompact.*

**PROOF.** Let  $\mathcal{U}$  be any open cover of  $X^*$  and choose  $U_\infty \in \mathcal{U}$  with  $\infty \in U$ . Then  $X^* - U_\infty \in \mathcal{F}$ , so we can find an open set  $V$  with  $X^* - U_\infty \subset V \subset X'$ ,  $\bar{V} \in \mathcal{F}$ . Now the sets of the form  $U \cap \bar{V}$ ,  $U \in \mathcal{U}$ , cover  $\bar{V}$ ; since  $\bar{V}$  is paracompact there is a locally finite refinement  $\mathcal{U}'$  of this cover of  $\bar{V}$ . Now let  $\mathcal{U}''$  be the family of all sets  $U' \cap V$ ,  $U' \in \mathcal{U}'$ . Then  $\gamma = \mathcal{U}'' \cup \{U_\infty\}$  is a locally finite open cover of  $X^*$ , refining  $\mathcal{U}$ .

To show that  $X^*$  is a Hausdorff space, let  $x$  and  $y$  be distinct points of  $X^*$ . If  $y = \infty$ , then  $\{x\} \in \mathcal{F}$ ; we choose a neighborhood  $U$  of  $x$  with  $\bar{U} \in \mathcal{F}$  and then  $U, X^* - \bar{U}$  are disjoint neighborhoods of  $x, y$ . If  $x \neq \infty$  and  $y \neq \infty$ , we choose  $U$  as before. If  $y \in \bar{U}$ , then  $U, X^* - \bar{U}$  are disjoint neighborhoods of  $x, y$ . If  $y \notin \bar{U}$ , then since  $\bar{U}$  is Hausdorff, we can choose open subsets  $V, W$  of  $X'$  with  $x \in V, y \in W, V \cap W \cap \bar{U} = \emptyset$ . Then  $V \cap U, W \cap (X^* - \bar{U})$  are disjoint neighborhoods of  $x, y$ .

Received by the editors October 24, 1958.

As examples of families of supports, we cite the following:

- (1) The compact subsets of a locally compact space;
- (2) The bounded closed subsets of a metric space;
- (3) The closed subsets of a space which do not meet a fixed subset  $A$ . This will be a phi-family if  $A$  is closed and the whole space  $X$  is paracompact.  $X^*$  is the quotient space of  $X$  obtained by identifying  $A$  to a point. Such a family may be used to define the relative cohomology  $H(X, A; \mathcal{Q})$  with coefficients in a sheaf  $\mathcal{Q}$ , in a manner described in the next section.

**3. Applications to sheaf theory.** Let  $\mathcal{F}$  be a family of supports for a space  $X$  and let  $\mathcal{Q}$  be a sheaf of rings or modules over  $X$ . We form the restriction  $\mathcal{Q}|X'$  of  $\mathcal{Q}$  to  $X'$  and then construct a sheaf  $\mathcal{Q}^*$  over  $X^*$  by defining  $\mathcal{Q}^*|X' = \mathcal{Q}|X'$  and letting the stalk  $A_\infty^*$  of  $\mathcal{Q}^*$  over  $\infty$  be zero. A neighborhood of  $A_\infty^*$  consists of the zeroes of stalks over a neighborhood of  $\infty$ . That the sheaf  $\mathcal{Q}^*$  is well determined by these data is easily verified. A map  $f: \mathcal{Q} \rightarrow \mathcal{B}$  of sheaves over  $X$  determines a map  $f^*: \mathcal{Q}^* \rightarrow \mathcal{B}^*$  of sheaves over  $X^*$  in an obvious way, and the following proposition is immediate.

**PROPOSITION 2.** *The correspondence  $\mathcal{Q} \rightarrow \mathcal{Q}^*, f \rightarrow f^*$  is an exact functor from the category of sheaves over  $X$  to that of sheaves over  $X^*$ .*

**PROPOSITION 3.** *If  $\mathcal{F}$  is a phi-family and if  $\mathcal{Q}$  is  $\mathcal{F}$ -fine, then  $\mathcal{Q}^*$  is fine.*

**PROOF.** Let  $\mathfrak{u}^*$  be a locally finite open cover of  $X^*$ . We may assume that  $\mathfrak{u}^* = \mathfrak{u}' \cup \{X^* - F\}$ , where  $\mathfrak{u}'$  is a cover of  $X'$  and  $F \in \mathcal{F}$ , since such covers are clearly cofinal among all covers of  $X^*$ . Then  $\mathfrak{u} = \mathfrak{u}' \cup \{X - F\}$  is a locally finite  $\mathcal{F}$ -cover of  $X$ . If  $\{l_i\}$  are endomorphisms of  $\mathcal{Q}$  making up a partition of unity for  $\mathcal{Q}$  and  $\mathfrak{u}$ , then the endomorphisms  $\{l_i^*\}$  make up a partition of unity for  $\mathcal{Q}^*$  and  $\mathfrak{u}^*$ . Hence  $\mathcal{Q}^*$  is fine.

**PROPOSITION 4.** *If  $\mathcal{F}$  is any family of supports on a space  $X$ , and if  $\mathcal{Q}$  is any sheaf over  $X$ , then there is a natural isomorphism*

$$\Gamma(X^*; \mathcal{Q}^*) \approx \Gamma_{\mathcal{F}}(X; \mathcal{Q}),$$

where  $\Gamma_{\mathcal{F}}(X; \mathcal{Q})$  is the module of sections of  $\mathcal{Q}$  over  $X$  with supports in  $\mathcal{F}$ .

The proof of Proposition 4 is immediate.

**THEOREM A.** *Let  $\mathcal{F}$  be a phi-family on a space  $X$ ,  $\mathcal{Q}$  a sheaf over  $X$ . Then there are natural isomorphisms, for each  $p \geq 0$ ,*

$$H^p(X^*; \mathcal{Q}^*) \approx H_{\mathcal{F}}^p(X; \mathcal{Q}).$$

PROOF. The preceding propositions shows that the modules  $H^p(X^*; \mathcal{A}^*)$  and their associated maps satisfy the Cartan axioms [3] for the  $\mathcal{F}$ -cohomology of  $\mathcal{A}$ .

We do not know whether or not the analog of Theorem A holds for arbitrary families of supports using the Grothendieck cohomology theory of sheaves [1; 2]. However, the following result is true.

THEOREM B. *If  $\mathcal{F}$  is any family of supports on a space  $X$  such that  $\cup \mathcal{F} = X$ , and if  $\mathcal{A}$  is any sheaf on  $X$ , then there are natural isomorphisms*

$$H^p(X^*; \mathcal{A}^*) \approx H_{\mathcal{F}}^p(X; \mathcal{A}),$$

where the cohomology modules are defined in the sense of Grothendieck.

The proof reduces to the easy verification that, under the assumption on  $\mathcal{F}$ , if  $\mathcal{A}$  is injective, then so is  $\mathcal{A}^*$ .

#### REFERENCES

1. R. Godement, *Theorie des faisceaux*, Paris, 1958.
2. A. Grothendieck, *Sur quelques points d'algebre homologique*, Tôhoku Math. J. (2) vol. 9 (1957) pp. 119-221.
3. *Seminaire Henri Cartan*, 1950-1951, 2d ed., Paris, 1955.

THE UNIVERSITY OF CHICAGO