ON PHI-FAMILIES
CHARLES E. WATTS

1. Introduction. The purpose of this note is to show that the notion of sections with support in a phi-family in the Cartan version of the Leray theory of sheaves can be avoided by the following expedient. One uses the phi-family to construct a new space in the manner of a one-point compactification. Then a sheaf on the original space is shown to yield a new sheaf on the new space whose cohomology (with unrestricted supports) is that of the original sheaf with restricted supports. A partial generalization to the Grothendieck cohomology theory is given.

2. Phi-families. A family \( \mathcal{F} \) of subsets of a topological space \( X \) is a family of supports [1] provided:

(I) each member of \( \mathcal{F} \) is closed;
(II) if \( FE\mathcal{F} \), then each closed subset of \( F \) is \( \in \mathcal{F} \);
(III) if \( F_1, F_2 \in \mathcal{F} \), then \( F_1 \cup F_2 \in \mathcal{F} \).

Given a family of supports \( \mathcal{F} \), we choose an object \( \infty \in X \), and define \( X' = U_{\mathcal{F}}, X* = X' \cup \{ \infty \} \). We then topologize \( X* \) by saying that a subset \( U \) of \( X* \) is open iff either \( U \) is open in \( X' \) or else \( X* - U \in \mathcal{F} \).

It is readily verified that these open sets in fact form a topology for \( X* \) and that the inclusion \( X' \subseteq X* \) is a topological imbedding.

The family of supports \( \mathcal{F} \) is a phi-family [3] provided also:

(IV) each member of \( \mathcal{F} \) has a closed neighborhood in \( \mathcal{F} \);
(V) each member of \( \mathcal{F} \) is paracompact.

Proposition 1. If \( \mathcal{F} \) is a phi-family, then \( X* \) is paracompact.

Proof. Let \( \mathcal{U} \) be any open cover of \( X* \) and choose \( U_\infty \in \mathcal{U} \) with \( \infty \in U \). Then \( X* - U_\infty \in \mathcal{F} \), so we can find an open set \( V \) with \( X* - U_\infty \subseteq V \subseteq X' \), \( V \in \mathcal{F} \). Now the sets of the form \( U \cap V, U \in \mathcal{U} \), cover \( V \); since \( V \) is paracompact there is a locally finite refinement \( \mathcal{U}' \) of this cover of \( V \). Now let \( \mathcal{U}'' \) be the family of all sets \( U' \cap V, U' \in \mathcal{U}' \). Then \( \gamma = \mathcal{U}'' \cup \{ U_\infty \} \) is a locally finite open cover of \( X* \), refining \( \mathcal{U} \).

To show that \( X* \) is a Hausdorff space, let \( x \) and \( y \) be distinct points of \( X* \). If \( y = \infty \), then \( \{ x \} \in \mathcal{F} \); we choose a neighborhood \( U \) of \( x \) with \( U \in \mathcal{F} \) and then \( U, X* - U \) are disjoint neighborhoods of \( x, y \). If \( x \neq \infty \) and \( y \neq \infty \), we choose \( U \) as before. If \( y \in U \), then \( U, X* - U \) are disjoint neighborhoods of \( x, y \). If \( y \in U \), then since \( U \) is Hausdorff, we can choose open subsets \( V, W \) of \( X' \) with \( x \in V, y \in W, V \cap W \cap U = \emptyset \). Then \( V \cap U, W \cap (X* - U) \) are disjoint neighborhoods of \( x, y \).

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As examples of families of supports, we cite the following:
(1) The compact subsets of a locally compact space;
(2) The bounded closed subsets of a metric space;
(3) The closed subsets of a space which do not meet a fixed subset \( A \). This will be a phi-family if \( A \) is closed and the whole space \( X \) is paracompact. \( X^* \) is the quotient space of \( X \) obtained by identifying \( A \) to a point. Such a family may be used to define the relative cohomology \( H(X, A; \mathfrak{a}) \) with coefficients in a sheaf \( \mathfrak{a} \), in a manner described in the next section.

3. Applications to sheaf theory. Let \( \mathcal{F} \) be a family of supports for a space \( X \) and let \( \mathfrak{a} \) be a sheaf of rings or modules over \( X \). We form the restriction \( \mathfrak{a}|_{X'} \) of \( \mathfrak{a} \) to \( X' \) and then construct a sheaf \( \mathfrak{a}^* \) over \( X^* \) by defining \( \mathfrak{a}^*|_{X'} = \mathfrak{a}|_{X'} \) and letting the stalk \( \mathfrak{a}^*_x \) of \( \mathfrak{a}^* \) over \( x \) be zero. A neighborhood of \( \mathfrak{a}^*_x \) consists of the zeroes of stalks over a neighborhood of \( x \). That the sheaf \( \mathfrak{a}^* \) is well determined by these data is easily verified. A map \( f: \mathfrak{a} \to \mathfrak{b} \) of sheaves over \( X \) determines a map \( f^*: \mathfrak{a}^* \to \mathfrak{b}^* \) of sheaves over \( X^* \) in an obvious way, and the following proposition is immediate.

Proposition 2. The correspondence \( \mathfrak{a} \to \mathfrak{a}^*, f \to f^* \) is an exact functor from the category of sheaves over \( X \) to that of sheaves over \( X^* \).

Proposition 3. If \( \mathcal{F} \) is a phi-family and if \( \mathfrak{a} \) is \( \mathcal{F} \)-fine, then \( \mathfrak{a}^* \) is fine.

Proof. Let \( \mathcal{U}^* \) be a locally finite open cover of \( X^* \). We may assume that \( \mathcal{U}^* = \mathcal{U}' \cup \{X^* - F\} \), where \( \mathcal{U}' \) is a cover of \( X' \) and \( F \subseteq \mathcal{F} \), since such covers are clearly cofinal among all covers of \( X^* \). Then \( \mathcal{U} = \mathcal{U}' \cup \{X - F\} \) is a locally finite \( \mathcal{F} \)-cover of \( X \). If \( \{l_i\} \) are endomorphisms of \( \mathfrak{a} \) making up a partition of unity for \( \mathfrak{a} \) and \( \mathcal{U} \), then the endomorphisms \( \{l_i^*\} \) make up a partition of unity for \( \mathfrak{a}^* \) and \( \mathcal{U}^* \). Hence \( \mathfrak{a}^* \) is fine.

Proposition 4. If \( \mathcal{F} \) is any family of supports on a space \( X \), and if \( \mathfrak{a} \) is any sheaf over \( X \), then there is a natural isomorphism
\[
\Gamma(X^*; \mathfrak{a}^*) \approx \Gamma_{\mathcal{F}}(X; \mathfrak{a}),
\]
where \( \Gamma_{\mathcal{F}}(X; \mathfrak{a}) \) is the module of sections of \( \mathfrak{a} \) over \( X \) with supports in \( \mathcal{F} \).

The proof of Proposition 4 is immediate.

Theorem A. Let \( \mathcal{F} \) be a phi-family on a space \( X \), \( \mathfrak{a} \) a sheaf over \( X \). Then there are natural isomorphisms, for each \( p \geq 0 \),
\[
H^p(X^*; \mathfrak{a}^*) \approx H^p_{\mathcal{F}}(X; \mathfrak{a}).
\]
Proof. The preceding propositions shows that the modules $H^p(X^*; \mathfrak{a}^*)$ and their associated maps satisfy the Cartan axioms [3] for the $\mathfrak{F}$-cohomology of $\mathfrak{a}$.

We do not know whether or not the analog of Theorem A holds for arbitrary families of supports using the Grothendieck cohomology theory of sheaves [1; 2]. However, the following result is true.

Theorem B. If $\mathfrak{F}$ is any family of supports on a space $X$ such that $\bigcup \mathfrak{F} = X$, and if $\mathfrak{a}$ is any sheaf on $X$, then there are natural isomorphisms

$$H^p(X^*; \mathfrak{a}^*) \cong H^p_{\mathfrak{F}}(X; \mathfrak{a}),$$

where the cohomology modules are defined in the sense of Grothendieck.

The proof reduces to the easy verification that, under the assumption on $\mathfrak{F}$, if $\mathfrak{a}$ is injective, then so is $\mathfrak{a}^*$.

References


The University of Chicago