ON PHI-FAMILIES

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1. **Introduction.** The purpose of this note is to show that the notion of sections with support in a phi-family in the Cartan version of the Leray theory of sheaves can be avoided by the following expedient. One uses the phi-family to construct a new space in the manner of a one-point compactification. Then a sheaf on the original space is shown to yield a new sheaf on the new space whose cohomology (with unrestricted supports) is that of the original sheaf with restricted supports. A partial generalization to the Grothendieck cohomology theory is given.

2. **Phi-families.** A family $\mathfrak{F}$ of subsets of a topological space X is a *family of supports* [1] provided:

(I) each member of $\mathfrak{F}$ is closed;

(II) if $F\in\mathfrak{F}$, then each closed subset of F is $\in\mathfrak{F}$;

(III) if $F_1, F_2\in\mathfrak{F}$, then $F_1\cup F_2\in\mathfrak{F}$.

Given a family of supports $\mathfrak{F}$, we choose an object $\infty\notin X$, and define $X'=\bigcup\mathfrak{F}$, $X^*=X'\cup\{\infty\}$. We then topologize X^* by saying that a subset U of X^* is open iff either U is open in X' or else $X^*-U\in\mathfrak{F}$. It is readily verified that these open sets in fact form a topology for X^* and that the inclusion $X'\subset X^*$ is a topological imbedding.

The family of supports $\mathfrak{F}$ is a *phi-family* [3] provided also:

(IV) each member of $\mathfrak{F}$ has a closed neighborhood in $\mathfrak{F}$;

(V) each member of $\mathfrak{F}$ is paracompact.

PROPOSITION 1. *If $\mathfrak{F}$ is a phi-family, then X^* is paracompact.*

PROOF. Let $\mathfrak{U}$ be any open cover of X^* and choose $U_\infty\in\mathfrak{U}$ with $\infty\in U$. Then $X^*-U_\infty\in\mathfrak{F}$, so we can find an open set V with $X^*-U_\infty\subset V\subset X'$, $\overline{V}\in\mathfrak{F}$. Now the sets of the form $U\cap\overline{V}$, $U\in\mathfrak{U}$, cover $\overline{V}$; since $\overline{V}$ is paracompact there is a locally finite refinement $\mathfrak{U}'$ of this cover of $\overline{V}$. Now let $\mathfrak{U}''$ be the family of all sets $U'\cap V$, $U'\in\mathfrak{U}'$. Then $\gamma=\mathfrak{U}''\cup\{U_\infty\}$ is a locally finite open cover of X^*, refining $\mathfrak{U}$.

To show that X^* is a Hausdorff space, let x and y be distinct points of X^*. If $y=\infty$, then $\{x\}\in\mathfrak{F}$; we choose a neighborhood U of x with $\overline{U}\in\mathfrak{F}$ and then U, $X^*-\overline{U}$ are disjoint neighborhoods of x, y. If $x\neq\infty$ and $y\neq\infty$, we choose U as before. If $y\notin\overline{U}$, then U, $X^*-\overline{U}$ are disjoint neighborhoods of x, y. If $y\in\overline{U}$, then since $\overline{U}$ is Hausdorff, we can choose open subsets V, W of X' with $x\in V$, $y\in W$, $V\cap W\cap\overline{U}=\emptyset$. Then $V\cap U$, $W\cap(X^*-\overline{U})$ are disjoint neighborhoods of x, y.

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As examples of families of supports, we cite the following:

(1) The compact subsets of a locally compact space;

(2) The bounded closed subsets of a metric space;

(3) The closed subsets of a space which do not meet a fixed subset A. This will be a phi-family if A is closed and the whole space X is paracompact. X^* is the quotient space of X obtained by identifying A to a point. Such a family may be used to define the relative cohomology $H(X, A; \mathfrak{A})$ with coefficients in a sheaf $\mathfrak{A}$, in a manner described in the next section.

3. **Applications to sheaf theory.** Let $\mathfrak{F}$ be a family of supports for a space X and let $\mathfrak{A}$ be a sheaf of rings or modules over X. We form the restriction $\mathfrak{A}|X'$ of $\mathfrak{A}$ to X' and then construct a sheaf $\mathfrak{A}^*$ over X^* by defining $\mathfrak{A}^*|X' = \mathfrak{A}|X'$ and letting the stalk A^*_∞ of $\mathfrak{A}^*$ over ∞ be zero. A neighborhood of A^*_∞ consists of the zeroes of stalks over a neighborhood of ∞. That the sheaf $\mathfrak{A}^*$ is well determined by these data is easily verified. A map f: $\mathfrak{A} \to \mathfrak{B}$ of sheaves over X determines a map f^*: $\mathfrak{A}^* \to \mathfrak{B}^*$ of sheaves over X^* in an obvious way, and the following proposition is immediate.

Proposition 2. *The correspondence $\mathfrak{A} \to \mathfrak{A}^*$, $f \to f^*$ is an exact functor from the category of sheaves over X to that of sheaves over X^*.*

Proposition 3. *If $\mathfrak{F}$ is a phi-family and if $\mathfrak{A}$ is $\mathfrak{F}$-fine, then $\mathfrak{A}^*$ is fine.*

Proof. Let $\mathfrak{U}^*$ be a locally finite open cover of X^*. We may assume that $\mathfrak{U}^* = \mathfrak{U}' \cup \{X^* - F\}$, where $\mathfrak{U}'$ is a cover of X' and $F \in \mathfrak{F}$, since such covers are clearly cofinal among all covers of X^*. Then $\mathfrak{U} = \mathfrak{U}' \cup \{X - F\}$ is a locally finite $\mathfrak{F}$-cover of X. If $\{l_i\}$ are endomorphisms of $\mathfrak{A}$ making up a partition of unity for $\mathfrak{A}$ and $\mathfrak{U}$, then the endomorphisms $\{l_i^*\}$ make up a partition of unity for $\mathfrak{A}^*$ and $\mathfrak{U}^*$. Hence $\mathfrak{A}^*$ is fine.

Proposition 4. *If $\mathfrak{F}$ is any family of supports on a space X, and if $\mathfrak{A}$ is any sheaf over X, then there is a natural isomorphism*

$$\Gamma(X^*; \mathfrak{A}^*) \approx \Gamma_{\mathfrak{F}}(X; \mathfrak{A}),$$

where $\Gamma_{\mathfrak{F}}(X; \mathfrak{A})$ is the module of sections of $\mathfrak{A}$ over X with supports in $\mathfrak{F}$.

The proof of Proposition 4 is immediate.

Theorem A. *Let $\mathfrak{F}$ be a phi-family on a space X, $\mathfrak{A}$ a sheaf over X. Then there are natural isomorphisms, for each $p \geqq 0$,*

$$H^p(X^*; \mathfrak{A}^*) \approx H^p_{\mathfrak{F}}(X; \mathfrak{A}).$$

PROOF. The preceding propositions shows that the modules $H^p(X^*;\ \mathfrak{A}^*)$ and their associated maps satisfy the Cartan axioms [**3**] for the $\mathfrak{F}$-cohomology of $\mathfrak{A}$.

We do not know whether or not the analog of Theorem A holds for arbitrary families of supports using the Grothendieck cohomology theory of sheaves [**1**; **2**]. However, the following result is true.

THEOREM B. *If $\mathfrak{F}$ is any family of supports on a space X such that $\cup\mathfrak{F}=X$, and if $\mathfrak{A}$ is any sheaf on X, then there are natural isomorphisms*

$$H^p(X^*;\ \mathfrak{A}^*) \approx H^p_{\mathfrak{F}}(X;\ \mathfrak{A}),$$

where the cohomology modules are defined in the sense of Grothendieck.

The proof reduces to the easy verification that, under the assumption on $\mathfrak{F}$, if $\mathfrak{A}$ is injective, then so is $\mathfrak{A}^*$.

REFERENCES

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