A QUESTION CONCERNING POSITIVE TYPE POLYNOMIALS

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1. **The problem.** In [1], the following problem is posed: Let $0<A<1$. Find the smallest integer n for which there exist n real numbers $a_1, a_2, \cdots, a_n$ such that the polynomial $(y^2-2Ay+1) \cdot \prod_{i=1}^{n}(y+a_i)$ has all its coefficients non-negative. If N is such an integer, does there exist some real number a such that $(y^2-2Ay+1) \cdot (y+a)^N$ has only non-negative coefficients.[1]

In the process of giving the solution, we recast the problem slightly and put it into relief against several similar problems.

Let $\mathcal{P}_{(n)}^{+}$ denote the class of positive type polynomials, [2], of degree n, that is, $\mathcal{P}_{(n)}^{+}=\{P_n(x)\}=\{\sum_{j=0}^{n} a_j x^j,\ a_j \geqq 0,\ j=0, 1, \cdots, n\}$. Here, we suppose, in addition, without loss of generality, that $a_0>0$, $a_n=1$. Define $\mathcal{P}^{+}=\mathcal{P}_0^{+}=\bigcup_{n=1}^{\infty}\mathcal{P}_{(n)}^{+}$. Further, let $\mathcal{P}_1^{+}$ (respectively, $\mathcal{P}_2^{+}$; $\mathcal{P}_3^{+}$) denote the class of polynomials, all of whose roots are real and negative (respectively, are equal and negative; have negative real part). What is the smallest integer $N_j = N_j(\epsilon)$ such that $(x^2-x+\epsilon)Q_{N_j}(x)\in\mathcal{P}^{+}$ where $\epsilon>1/4$ and $Q_{N_j}(x)$ is a polynomial of degree N_j which belongs to $\mathcal{P}_j^{+}$? Here, j takes any of the values 0, 1, 2, 3. If $j=0$, the answer is contained in the statement and proof of Theorem 1 of [2]. For $j=1$, the smallest integer n (namely N_1) for which $(x^2-x+\epsilon)\prod_{i=1}^{n}(x+b_i)\in\mathcal{P}^{+}$ is exactly the integer N mentioned earlier, as the correspondence $y\to 2Ax$, $\epsilon\to(4A^2)^{-1}$, $a_i\to Ab_i$ shows. Clearly, $b_i\geqq 0$, all i and when $n=N_1$, $b_i>0$, $i=1, 2, \cdots, N_1$ For $j=2$, the above question is intimately related to the second part of the research problem while for $j=3$, the connection is more remote.

All of these questions are rendered easier by an inversion, that is, the fixing of n and the quest for the smallest $\epsilon=\epsilon_n$ such that

$$(x^2 - x + \epsilon)Q_n(x) \in \mathcal{P}^{+}. \tag{1.1}$$

It is then a simple matter to express conditions for (1.1) in terms of the coefficients of $Q_n(x)$. The difficulties arise in fulfilling these and simultaneously insuring that $Q_n(x)\in\mathcal{P}_j^{+}$. Fortunately, when $j=1$ (which will henceforth be supposed), necessary conditions that

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[1] A third part of the problem asks for a generalization in which the factor $(y^2-2Ay+1)$ is replaced by an arbitrary polynomial (without positive roots). This appears to be of a different order of difficulty.

$Q_n(x)=\prod_{i=1}^n (x+b_i)\in\mathcal{P}_1^+$ are sufficient to resolve the problem. Writing $b=(b_1,\cdots,b_n)$ any pair (ϵ, b) satisfying (1.1) will be called a "real solution." $C_{n,j}$ will denote the combinatorial symbol.

2. **The solution.** Define $Q_n(x)=\sum_{j=0}^n C_{n,j}p_jx^{n-j}$, $p_0=1$. For (1.1) to hold, it is evidently necessary and sufficient that

$$C_{n,j+1}p_{j+1}-C_{n,j}p_j+\epsilon C_{n,j-1}p_{j-1}\geqq 0,\qquad j=1,\cdots,n-1, \tag{2.1}$$

$$np_1\geqq 1,\qquad \epsilon np_{n-1}\geqq p_n. \tag{2.1$'$}$$

Define further $r_j=p_j/p_{j-1}$, $j=1,2,\cdots,n$. The preceding equations take the form

$$\frac{n-j}{j+1}r_{j+1}+\frac{\epsilon j}{n-j+1}\frac{1}{r_j}\geqq 1,\qquad j=1,2,\cdots,n-1, \tag{2.2}$$

$$r_1\geqq n^{-1},\qquad r_n\leqq \epsilon n. \tag{2.2$'$}$$

A necessary condition, [3], that in addition, $Q_n(x)=\prod_{i=1}^n(x+b_i)$ for real b_i, that is, $p_j=(C_{n,j})^{-1}\sum b_{i_1}b_{i_2}\cdots b_{i_j}$ where the summation is over indices $1\leqq i_1<i_2<\cdots<i_j\leqq n$, is conveniently expressed as

$$r_1\geqq r_2\geqq\cdots\geqq r_n. \tag{2.3}$$

The equality of any pair of the r's, say, $r_i=r_j=r$ implies the equality of all intervening r's (if any) and hence $b_1=b_2=\cdots=b_n=r$. Conversely, $b_1=b_2=\cdots=b_n=b$ clearly implies $r_1=r_2=\cdots=r_n=b$. These facts will be used later.

It follows directly from (2.2) and (2.3) that any real solution must satisfy also

$$r_{j+1}^2-\frac{j+1}{n-j}r_{j+1}+\frac{\epsilon j(j+1)}{(n-j)(n-j+1)}\geqq 0,\ j=1,\cdots,n-1; \tag{2.4}$$

that is, for $j=1,2,\cdots,n-1$

$$\begin{aligned} r_{j+1}&\leqq\frac{j+1}{2(n-j)}\left[1-\left(1-\frac{4\epsilon j(n-j)}{(j+1)(n-j+1)}\right)^{1/2}\right]\\ &=\bar{r}_{j+1}(\epsilon)\qquad\text{(say)}\end{aligned} \tag{2.5}$$

or

$$r_{j+1}\geqq\frac{j+1}{2(n-j)}\left[1+\left(1-\frac{4\epsilon j(n-j)}{(j+1)(n-j+1)}\right)^{1/2}\right]=\overset{+}{r}_{j+1}(\epsilon)\quad\text{(say)}.$$

We pause now in the general line of reasoning to establish two crucial results. Let ϵ be such that $r_j^-=r_j^-(\epsilon)$ and $r_j^+=r_j^+(\epsilon)$ are real valued for $j=2,3,\cdots,n$:

A. (a) If n is odd and $\epsilon \leqq (n+3)^2/4(n+1)^2$ then $r_{j+1}^-(\epsilon) \leqq r_j^+(\epsilon)$ for $j=1, 2, \cdots, n-1$. Further, for this range of ϵ, equality holds only if $j=(n+1)/2$ and $\epsilon=(n+3)^2/4(n+1)^2$.

(b) If n is even and $\epsilon \leqq (n+4)/4n$ then $r_{j+1}^-(\epsilon) \leqq r_j^+(\epsilon)$ for $j=1, 2, \cdots, n-1$. Further, for this range of ϵ, equality holds only if $j=n/2$ or $(n+2)/2$, and $\epsilon=(n+4)/4n$.

PROOF. $r_{j+1}^- \leqq r_j^+$ is tantamount to

$$\frac{(j+1)(n-j+1)-((j+1)^2(n-j+1)^2-4\epsilon j(i+1)(n-j)(n-j+1))^{1/2}}{2(n-j)(n-j+1)}$$
$$\leqq \frac{j(n-j+2)+(j^2(n-j+2)^2-4\epsilon j(j-1)(n-j+1)(n-j+2))^{1/2}}{2(n-j+1)(n-j+2)},$$

or

$$(n+1)-((j+1)^2(n-j+1)^2-4\epsilon j(j+1)(n-j)(n-j+1))^{1/2}$$
$$\leqq (n-j)(j^2-4\epsilon j(j-1)(n-j+1)(n-j+2)^{-1})^{1/2}.$$

If j is such that the left hand side (L.H.S.) is negative, the preceding is automatically true; otherwise it is equivalent to

$$4\epsilon j(n-j+1)-j(n-j+2)$$
$$\leqq (j^2(n-j+2)^2-4\epsilon j(j-1)(n-j+1)(n-j+2))^{1/2}.$$

Again, this is trivially true if the L.H.S. is negative and otherwise equivalent to $4\epsilon j(n-j+1) \leqq (j+1)(n-j+2)$ or

$$j(n-j+1) \leqq (n+2)(4\epsilon-1)^{-1}. \tag{2.6}$$

If now n is odd, rewrite (2.6) in the form

$$0 \leqq \left(j-\frac{n+1}{2}\right)^2+\frac{n+2}{4\epsilon-1}-\frac{(n+1)^2}{4}$$

and, retracing the steps, this yields (a).

On the other hand if n is even, rewrite (2.6) as

$$0 \leqq \left(j-\frac{n}{2}\right)\left(j-\frac{n+2}{2}\right)+(n+2)\left[\frac{1}{4\epsilon-1}-\frac{n}{4}\right]$$

from which (b) is deduced.

B. (a) If $r_{j+1} \leqq r_{j+1}^-$, then $r_j \leqq r_{j+1}^-$, $j=1, 2 \cdots n-1$.

(b) If, for n odd, $\epsilon \leqq (n+3)^2/4(n+1)^2$ or for n even, $\epsilon \leqq (n+4)/4n$ then $r_j \geqq r_j^+$ implies $r_{j+1} \geqq r_j^-$, $j=1, 2 \cdots n-1$ (equality for n odd only if $j=(n+1)/2$ and $\epsilon=(n+3)^2/4(n+1)^2$; equality for n even only if $j=n/2$ or $(n+2)/2$, and $\epsilon=(n+4)/4n$).

PROOF: Noting from (2.4) that

$$(2.7)\qquad \overset{-}{r}_{j+1}\cdot\overset{+}{r}_{j+1}=\frac{\epsilon j(j+1)}{(n-j)(n-j+1)},\qquad j=1,2\cdots n-1,$$

we see from (2.2) and the hypothesis of (a) that

$$\begin{aligned} r_j &\leqq \epsilon j(n-j+1)^{-1}\left[1-\frac{n-j}{j+1}\overset{-}{r}_{j+1}\right]^{-1} \\ &= \epsilon j(n-j+1)^{-1}\left[\frac{1}{2}\left(1+\left(1-\frac{4\epsilon j(n-j)}{(j+1)(n-j+1)}\right)^{1/2}\right)\right]^{-1} \\ &= \frac{j+1}{2(n-j)}\left[1-\left(1-\frac{4\epsilon j(n-j)}{(j+1)(n-j+1)}\right)^{1/2}\right]=\overset{-}{r}_{j+1}. \end{aligned}$$

To prove (b), we note from A that $\epsilon j/(n-j+1)\cdot 1/r_{j+1}^{-}>\epsilon j/(n-j+1)\cdot 1/r_j^{+}$, or equivalently

$$\begin{aligned} 1-\frac{\epsilon j}{(n-j+1)r_j^{+}} &> 1-\frac{1}{2}\left[1+\left(1-\frac{4\epsilon j(n-j)}{(j+1)(n-j+1)}\right)^{1/2}\right] \\ &= \frac{n-j}{j+1}\overset{-}{r}_{j+1}. \end{aligned}$$

Hence, the hypothesis of (b) and (2.2) imply

$$\frac{n-j}{j+1}r_{j+1}\geqq 1-\frac{\epsilon j}{(n-j+1)r_j^{+}}>\frac{n-j}{j+1}\overset{-}{r}_{j+1}$$

which yields the conclusion of (b).

In returning to the problem as a whole, we discard the possibility that $r_j^{-}(\epsilon)$ and $r_j^{+}(\epsilon)$ are complex valued since this occurs only for values of ϵ larger than those for which we establish solutions. Also, we must separate the case of odd n from that of even n. Suppose first that $n=2k+1$ where k is a non-negative integer. We shall show that $\epsilon_n'=(n+3)^2/4(n+1)^2$ is the smallest value of ϵ for which there exist real numbers $r_1, r_2\cdots r_n$ satisfying (2.2), (2.2)$'$ and (2.3). As a consequence, no real solution is possible for $\epsilon<\epsilon_n'$. We shall also find a (unique) real solution for $\epsilon=\epsilon_n'$. To this end, we distinguish, via (2.5), three mutually exclusive and exhaustive possibilities:

(i) $r_{k+1}\geqq r_{k+1}^{+}(\epsilon)$, $r_{k+2}\leqq r_{k+2}^{-}(\epsilon)$,

(ii) $r_{k+1}\leqq r_{k+1}^{-}(\epsilon)$,

(iii) $r_{k+2}\geqq r_{k+2}^{+}(\epsilon)$.

If (i) holds, (2.2) and (i) imply

$$\bar{r}_{k+2} \geqq r_{k+2} \geqq \frac{k+2}{k}\left[1 - \frac{\epsilon}{r^+_{k+1}}\right]. \tag{2.8}$$

Substituting (2.7) for $j=k$ into (2.8) and recalling the definitions of r^+_{k+1}, r^-_{k+2}, we find that

$$\begin{aligned} &\frac{k+2}{2k}\left[1 - (1 - 4\epsilon k(k+2)^{-1})^{1/2}\right] \\ &\geqq r_{k+2} \geqq \frac{k+2}{k}\left[1 - \frac{k+2}{2k}\left(1 - (1 - 4\epsilon k(k+2)^{-1})^{1/2}\right)\right] \end{aligned} \tag{2.9}$$

which is valid only if $\epsilon \geqq (k+2)^2/4(k+1)^2 = (n+3)^2/4(n+1)^2 = \epsilon_n'$. We now show that a real solution does exist for $\epsilon = \epsilon_n'$. When $\epsilon = \epsilon_n'$, (2.9) yields $r_{k+2} = (k+2)/2(k+1)$. Substituting this value of r_{k+2} and $\epsilon = \epsilon_n'$ back in (2.2), we find $r_{k+1} \leqq (k+2)/2(k+1)$ which combined with (i) $r_{k+1} \geqq r^+_{k+1}(\epsilon_n') = (k+2)/2(k+1)$ implies

$$r_{k+1} = (k+2)/2(k+1) = r_{k+2}.$$

But then, as remarked earlier, $b_j = (k+2)/2(k+1) = (n+3)/2(n+1) = \bar{b}_n'$ (say), $j = 1, 2 \cdots n$ implying $r_j = (n+3)/2(n+1)$, $j = 1, 2, \cdots n$ and these values of r_j are quite compatible with (2.2), (2.2)' and (2.3). Consequently,[2] $(\epsilon_n', \bar{b}_n')$ is a real solution and among all real solutions conforming to (i), it is the one having minimal ϵ.

Since we are concerned only with the possibility of solutions with smaller ϵ, we assume in what follows that $\epsilon \leqq \epsilon_n'$. Suppose next that (ii) $r_{k+1} \leqq r^-_{k+1}$ prevails. Part (a) of B yields $r_k \leqq r^-_{k+1}$. But then necessarily $r_k \leqq r_k^-$ since the only alternative permitted by (2.5), namely $r_k \geqq r_k^+$ entails (by proposition A) $r_k \geqq r_k^+ > r^-_{k+1}$, a contradiction. Repeating this chain of argument, we eventually deduce $r_2 \leqq r_2^-$. Applying (a) of B once more, we find $r_1 \leqq r_2^- = [1/n-1][1-(1-2\epsilon_n'(n-1)n^{-1})^{1/2}]$ which contradicts (2.2)' for $n \geqq 3$.

Finally, the contingency (iii) $r_{k+2} \geqq r^+_{k+2}$ implies by (b) of B that $r_{k+3} > r^-_{k+3}$. But then (2.5) insures that $r_{k+3} \geqq r^+_{k+3}$. Continuing in this fashion, we find that $r_n \geqq r_n^+ > n\epsilon$ (for $\epsilon \leqq \epsilon_n'$) contradicting (2.2)' for $n > 1$.

When $n=1$, it is trivial to verify that $\epsilon = \epsilon_1' = 1$, $b = \bar{b}_1' = 1$ is the unique minimal real solution.

Thus, when $n = 2k+1$, $P_{n+2}(x) = (x^2 - x + \epsilon)\prod_{i=1}^n (x + b_i) \in \mathcal{P}^+$ only if $\epsilon \geqq \epsilon_n'$. Further, when $\epsilon = \epsilon_n'$ there is only one possible choice of b, namely, $b_1 = b_2 = \cdots = b_n = (\epsilon_n')^{1/2}$. These values cause the central terms of $P_{n+2}(x)$ of degree $k+1$ and $k+2$ to vanish.

[2] It is to be understood wherever a real solution is concerned that (ϵ, z) abbreviates the $(n+1)$-tuple $(\epsilon, z, z \cdots z)$.

We turn now to the case $n=2k$, k a positive integer, and distinguish four mutually exclusive and exhaustive possibilities:

(i) $r_k \geqq r_k^+(\epsilon)$, $r_{k+1} \leqq r_{k+1}^-(\epsilon)$;

(ii) $r_{k+1} \geqq r_{k+1}^+(\epsilon)$, $r_{k+2} \leqq r_{k+2}^-(\epsilon)$;

(iii) $r_k \leqq r_k^-(\epsilon)$,

(iv) $r_{k+2} \geqq r_{k+2}^+(\epsilon)$.

Under case (i), employing (2.2) and then (2.7) for $j=k$, we have

$$
\begin{aligned}
r_{k+1}^-(\epsilon) \geqq r_{k+1} &\geqq \frac{k+1}{k}\left[1 - \frac{\epsilon k}{(k+1)r_k^+}\right] \\
&= \frac{k+1}{k}\left[1 - \frac{k+2}{k-1}\, r_k^-\right]
\end{aligned}
\tag{2.10}
$$

which is valid only when $\epsilon \geqq (k+2)/4k = (n+4)/4n = \epsilon_n''$ (say). We show next that a solution exists for $\epsilon = \epsilon_n''$.

For this determination of ϵ, (2.10) implies that $r_{k+1} = 1/2$. Substituting these values back in (2.2) we find $r_k \leqq 1/2$, which, combined with (i) $r_k \geqq r_k^+(\epsilon_n'') = 1/2$, yields the result $r_k = 1/2 = r_{k+1}$. But then $b_j \equiv 1/2$ which, in turn, forces $r_j \equiv 1/2$ and it is readily checked that these latter values satisfy (2.2) and (2.2)$'$. Thus,[2] $(\epsilon_n'', 1/2)$ is a real solution and no other real solution compatible with (i) permits a smaller choice of ϵ.

In case (ii) $r_{k+1} \geqq r_{k+1}^+(\epsilon)$, $r_{k+2} \leqq r_{k+2}^-(\epsilon)$, analogous reasoning again yields $\epsilon \geqq \epsilon_n''$ but now $r_{k+1} = r_{k+2} = (k+2)/2k = (n+4)/2n = 2\epsilon_n''$ and it is easily verified that $(\epsilon_n'', 2\epsilon_n'')$ also constitutes a real solution.[2]

The remaining cases (iii) $r_k \leqq r_k^-(\epsilon)$ and (iv) $r_{k+2} \geqq r_{k+2}^+(\epsilon)$ may be disposed of, as in the case n odd, by a systematic use of propositions A and B.

Thus, when $n=2k$, $P_{n+2}(x) = (x^2 - x + \epsilon)\prod_{i=1}^n (x+b_i) \in \mathcal{P}^+$ only if $\epsilon \geqq \epsilon_n''$ and when $\epsilon = \epsilon_n''$, there are two real solutions obtained by choosing $b_j \equiv 2\epsilon_n''$ or $b_j \equiv 1/2$. In the former case the coefficients of the terms of degree k and $k+1$ in $P_{n+2}(x)$ are zero while in the latter case the terms x^k and x^{k-1} vanish. Note that $1/2 < 2\epsilon_n'' \leqq 1$ and that $2\epsilon_n'' \searrow 1/2$. Similarly, $1/2 < \bar{b}_n' = (\epsilon_n')^{1/2} \leqq 1$ and $\bar{b}_n' \searrow 1/2$.

Define

$$
\begin{aligned}
\epsilon_n &= \epsilon_n', \; n \text{ odd} & \bar{\bar{b}}_n = \bar{b}_n &= (\epsilon_n')^{1/2}, \; n \text{ odd}, \\
&= \epsilon_n'', \; n \text{ even} & \bar{b}_n &= 2\epsilon_n'', \; n \text{ even}, \\
& & \bar{\bar{b}}_n &= 1/2, \; n \text{ even}.
\end{aligned}
$$

Since $\epsilon_n' > \epsilon_{n+1}''$ and $\epsilon_n'' > \epsilon_{n+1}'$, the sequence $\{\epsilon_n\}$ is monotone decreasing (to 1/4). Hence, the step function (take $\epsilon_0 = +\infty$)

$$
N_1(\epsilon) = j, \qquad \epsilon_j \leqq \epsilon < \epsilon_{j-1}, \qquad\qquad j = 1, 2 \cdots
$$

gives the answer to the question initially posed.

Clearly, if $\epsilon' > \epsilon$ and $P_{n+2}(x;\epsilon) = (x^2 - x + \epsilon)\prod_{i=1}^{n}(x + b_i) \in \mathcal{P}^+$, then $P_{n+2}(x;\epsilon') = P_{n+2}(x;\epsilon) + (\epsilon' - \epsilon)\prod_{i=1}^{n}(x + b_i) \in \mathcal{P}^+$. Consequently, $(x^2 - x + \epsilon)(x + \bar{b}_n)^n \in \mathcal{P}^+$ for all $\epsilon \geqq \epsilon_n$ and likewise $(x^2 - x + \epsilon)(x + \bar{b}_n)^n \in \mathcal{P}^+$, $\epsilon \geqq \epsilon_n$. Thus, the answer to the second question raised is affirmative. The real numbers $b_1, b_2 \cdots b_n$ may always be selected equal without increasing n and indeed when $\epsilon = \epsilon_n$, they must be chosen equal if n is to be kept minimal. Expressed otherwise, $N_1(\epsilon) = N_2(\epsilon)$.

3. **Remarks.** In this section, we give a more precise sufficient condition than that contained in Theorem 1 of [2] which states that a necessary and sufficient condition that there exist $P_m(x) = \sum_{j=0}^{m} a_j x^j = (x^2 - x + \epsilon)\, P_{m-2}(x) \in \mathcal{P}^+$ is that $m \geqq M(\epsilon)$ where $M(\epsilon)$ is equal to the smallest integer j for which j arc cos $(2(\epsilon)^{1/2})^{-1} \geqq \pi$. Here, $m-2$ and $M(\epsilon) - 2$ correspond respectively to the n and $N_1(\epsilon)$ of §1. We employ the notation and results of [2].

Analogous to (2.1) (of §2 or of [2]), we have (take $P_{m-2}(x) = \sum_{j=0}^{m-2} c_j x^j$)

$$c_{j-2} - c_{j-1} + \epsilon c_j = a_j \geqq 0, \qquad j = 0, 1 \cdots m \tag{3.1}$$

as a necessary and sufficient condition that $P_m(x) \in \mathcal{P}^+$. Here, $c_{-2} = c_{-1} = c_{m-1} = c_m = 1 - c_{m-2} = 0$.

Multiply (3.1) by $\bar{g}_{j-1} = \bar{g}_{j-1}(\epsilon)$ (see [2] for definition) and sum from $j=2$ to $j=i$ obtaining (via (1.6) of [2] and $\bar{g}_1 = \bar{g}_2 = 1$)

$$c_0 - (\bar{g}_{i-1} - \epsilon\bar{g}_{i-2}) + \epsilon\bar{g}_{i-1}c_i = \sum_{j=2}^{i} a_j\bar{g}_{j+1}. \tag{3.2}$$

Hence, from (1.3) of [2],

$$c_i \geqq \frac{\bar{g}_i(\epsilon)c_{i-1} - c_0}{\epsilon\bar{g}_{i-1}(\epsilon)} \qquad \text{for } i < M(\epsilon) - 1. \tag{3.3}$$

In analogous fashion, multiply (3.1) by $\epsilon^{-(m-2-j)}\bar{g}_{m-1-j}$ and sum from $j = m-2$ to $j = m-i$ to obtain

$$\begin{aligned} \epsilon - \epsilon^{-(i-2)}[\bar{g}_{i-1} - \epsilon\bar{g}_{i-2}]c_{m-i-1} + \epsilon^{-(i-2)}\bar{g}_{i-1}c_{m-i-2} \\ = \sum_{r=2}^{i} \epsilon^{-(r-2)}\bar{g}_{r-1}a_{m-r}, \end{aligned} \tag{3.4}$$

implying

$$c_j \geqq \frac{\bar{g}_{m-j-2}c_{j+1} - \epsilon^{m-j-3}}{\bar{g}_{m-j-3}} \quad \text{for } m - j < M(\epsilon) - 1. \tag{3.5}$$

Now, the trigonometric representation (see (1.3) of [2]) of $g_j(\delta)$ readily yields the identity $g_j^2(\delta)-(1+\delta)^{j-1}=g_{j-1}(\delta)g_{j+1}(\delta)$ or equivalently

$$\bar{g}_j^2(\epsilon) - \epsilon^{j-1} = \bar{g}_{j-1}(\epsilon)\bar{g}_{j+1}(\epsilon), \qquad j \geqq 2. \tag{3.6}$$

Then, utilizing (3.6), an induction on (3.3) gives

$$c_j \geqq \epsilon^{-j}\bar{g}_{j+1}(\epsilon)c_0, \qquad 1 \leqq j < M(\epsilon) - 1. \tag{3.7}$$

Similarly, induction on (3.5) leads to

$$c_{m-j} \geqq \bar{g}_{j-1}(\epsilon), \qquad m - j < M(\epsilon) - 1. \tag{3.8}$$

Thus, (3.7) and (3.8) furnish simple necessary conditions for the coefficients c_j in terms of the known polynomials $\bar{g}_j(\epsilon)$.

Finally, we note that choosing c_j equal to the lower bound of (3.8) gives

$$\begin{aligned} P_m(x) = (x^2 - x + \epsilon)\sum_{i=0}^{m-2}\bar{g}_{m-i-1}(\epsilon)x^i &= x^m + (\epsilon\bar{g}_{m-2} - \bar{g}_{m-1})x + \epsilon\bar{g}_{m-1}(\epsilon) \\ &= x^m - \bar{g}_m(\epsilon)x + \epsilon\bar{g}_{m-1}(\epsilon). \end{aligned}$$

But for $m=M(\epsilon)$ (as well as for a range of larger values), $g_m(\epsilon)\leqq 0$ whence $P_m(x)\in\mathcal{P}^+$, $m=M(\epsilon)$.

A trinomial of this form was constructed in the sufficiency proof of Theorem 1 of [2] but the preceding gives the nonvanishing coefficients as explicit functions of ϵ and at the same time specifies the coefficients of $(x^2-x+\epsilon)^{-1}P_m(x)$.

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