

A QUESTION CONCERNING POSITIVE TYPE POLYNOMIALS

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1. **The problem.** In [1], the following problem is posed: Let $0 < A < 1$. Find the smallest integer n for which there exist n real numbers a_1, a_2, \dots, a_n such that the polynomial $(y^2 - 2Ay + 1) \cdot \prod_{i=1}^n (y + a_i)$ has all its coefficients non-negative. If N is such an integer, does there exist some real number a such that $(y^2 - 2Ay + 1) \cdot (y + a)^N$ has only non-negative coefficients.¹

In the process of giving the solution, we recast the problem slightly and put it into relief against several similar problems.

Let \mathcal{P}_n^+ denote the class of positive type polynomials, [2], of degree n , that is, $\mathcal{P}_n^+ = \{P_n(x)\} = \{\sum_{j=0}^n a_j x^j, a_j \geq 0, j=0, 1, \dots, n\}$. Here, we suppose, in addition, without loss of generality, that $a_0 > 0, a_n = 1$. Define $\mathcal{P}^+ = \mathcal{P}_0^+ = \bigcup_{n=1}^{\infty} \mathcal{P}_n^+$. Further, let \mathcal{P}_1^+ (respectively, \mathcal{P}_2^+ ; \mathcal{P}_3^+) denote the class of polynomials, all of whose roots are real and negative (respectively, are equal and negative; have negative real part). What is the smallest integer $N_j = N_j(\epsilon)$ such that $(x^2 - x + \epsilon)Q_{N_j}(x) \in \mathcal{P}^+$ where $\epsilon > 1/4$ and $Q_{N_j}(x)$ is a polynomial of degree N_j which belongs to \mathcal{P}_j^+ ? Here, j takes any of the values 0, 1, 2, 3. If $j=0$, the answer is contained in the statement and proof of Theorem 1 of [2]. For $j=1$, the smallest integer n (namely N_1) for which $(x^2 - x + \epsilon) \prod_{i=1}^n (x + b_i) \in \mathcal{P}^+$ is exactly the integer N mentioned earlier, as the correspondence $y \rightarrow 2Ax, \epsilon \rightarrow (4A^2)^{-1}, a_i \rightarrow Ab_i$ shows. Clearly, $b_i \geq 0$, all i and when $n = N_1, b_i > 0, i = 1, 2, \dots, N_1$. For $j=2$, the above question is intimately related to the second part of the research problem while for $j=3$, the connection is more remote.

All of these questions are rendered easier by an inversion, that is, the fixing of n and the quest for the smallest $\epsilon = \epsilon_n$ such that

$$(1.1) \quad (x^2 - x + \epsilon)Q_n(x) \in \mathcal{P}^+.$$

It is then a simple matter to express conditions for (1.1) in terms of the coefficients of $Q_n(x)$. The difficulties arise in fulfilling these and simultaneously insuring that $Q_n(x) \in \mathcal{P}_j^+$. Fortunately, when $j=1$ (which will henceforth be supposed), necessary conditions that

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¹ A third part of the problem asks for a generalization in which the factor $(y^2 - 2Ay + 1)$ is replaced by an arbitrary polynomial (without positive roots). This appears to be of a different order of difficulty.

$Q_n(x) = \prod_{i=1}^n (x + b_i) \in \mathcal{P}_1^+$ are sufficient to resolve the problem. Writing $b = (b_1, \dots, b_n)$ any pair (ϵ, b) satisfying (1.1) will be called a "real solution." $C_{n,j}$ will denote the combinatorial symbol.

2. The solution. Define $Q_n(x) = \sum_{j=0}^n C_{n,j} p_j x^{n-j}$, $p_0 = 1$. For (1.1) to hold, it is evidently necessary and sufficient that

$$(2.1) \quad C_{n,j+1} p_{j+1} - C_{n,j} p_j + \epsilon C_{n,j-1} p_{j-1} \geq 0, \quad j = 1, \dots, n-1,$$

$$(2.1)' \quad n p_1 \geq 1, \quad \epsilon n p_{n-1} \geq p_n.$$

Define further $r_j = p_j / p_{j-1}$, $j = 1, 2, \dots, n$. The preceding equations take the form

$$(2.2) \quad \frac{n-j}{j+1} r_{j+1} + \frac{\epsilon j}{n-j+1} \frac{1}{r_j} \geq 1, \quad j = 1, 2, \dots, n-1,$$

$$(2.2)' \quad r_1 \geq n^{-1}, \quad r_n \leq \epsilon n.$$

A necessary condition, [3], that in addition, $Q_n(x) = \prod_{i=1}^n (x + b_i)$ for real b_i , that is, $p_j = (C_{n,j})^{-1} \sum b_{i_1} b_{i_2} \dots b_{i_j}$ where the summation is over indices $1 \leq i_1 < i_2 < \dots < i_j \leq n$, is conveniently expressed as

$$(2.3) \quad r_1 \geq r_2 \geq \dots \geq r_n.$$

The equality of any pair of the r 's, say, $r_i = r_j = r$ implies the equality of all intervening r 's (if any) and hence $b_1 = b_2 = \dots = b_n = r$. Conversely, $b_1 = b_2 = \dots = b_n = b$ clearly implies $r_1 = r_2 = \dots = r_n = b$. These facts will be used later.

It follows directly from (2.2) and (2.3) that any real solution must satisfy also

$$(2.4) \quad r_{j+1}^2 - \frac{j+1}{n-j} r_{j+1} + \frac{\epsilon j(j+1)}{(n-j)(n-j+1)} \geq 0, \quad j = 1, \dots, n-1;$$

that is, for $j = 1, 2, \dots, n-1$

$$(2.5) \quad r_{j+1} \leq \frac{j+1}{2(n-j)} \left[1 - \left(1 - \frac{4\epsilon j(n-j)}{(j+1)(n-j+1)} \right)^{1/2} \right] \\ = \bar{r}_{j+1}(\epsilon) \quad (\text{say})$$

or

$$r_{j+1} \geq \frac{j+1}{2(n-j)} \left[1 + \left(1 - \frac{4\epsilon j(n-j)}{(j+1)(n-j+1)} \right)^{1/2} \right] = r_{j+1}^+(\epsilon) \quad (\text{say}).$$

We pause now in the general line of reasoning to establish two crucial results. Let ϵ be such that $r_j^- = r_j^-(\epsilon)$ and $r_j^+ = r_j^+(\epsilon)$ are real valued for $j = 2, 3, \dots, n$:

A. (a) If n is odd and $\epsilon \leq (n+3)^2/4(n+1)^2$ then $r_{\bar{j}+1}(\epsilon) \leq r_j^+(\epsilon)$ for $j=1, 2, \dots, n-1$. Further, for this range of ϵ , equality holds only if $j=(n+1)/2$ and $\epsilon=(n+3)^2/4(n+1)^2$.

(b) If n is even and $\epsilon \leq (n+4)/4n$ then $r_{\bar{j}+1}(\epsilon) \leq r_j^+(\epsilon)$ for $j=1, 2, \dots, n-1$. Further, for this range of ϵ , equality holds only if $j=n/2$ or $(n+2)/2$, and $\epsilon=(n+4)/4n$.

PROOF. $r_{\bar{j}+1} \leq r_j^+$ is tantamount to

$$\frac{(j+1)(n-j+1) - ((j+1)^2(n-j+1)^2 - 4\epsilon j(j+1)(n-j)(n-j+1))^{1/2}}{2(n-j)(n-j+1)} \leq \frac{j(n-j+2) + (j^2(n-j+2)^2 - 4\epsilon j(j-1)(n-j+1)(n-j+2))^{1/2}}{2(n-j+1)(n-j+2)},$$

or

$$(n+1) - ((j+1)^2(n-j+1)^2 - 4\epsilon j(j+1)(n-j)(n-j+1))^{1/2} \leq (n-j)(j^2 - 4\epsilon j(j-1)(n-j+1)(n-j+2)^{-1})^{1/2}.$$

If j is such that the left hand side (L.H.S.) is negative, the preceding is automatically true; otherwise it is equivalent to

$$4\epsilon j(n-j+1) - j(n-j+2) \leq (j^2(n-j+2)^2 - 4\epsilon j(j-1)(n-j+1)(n-j+2))^{1/2}.$$

Again, this is trivially true if the L.H.S. is negative and otherwise equivalent to $4\epsilon j(n-j+1) \leq (j+1)(n-j+2)$ or

$$(2.6) \quad j(n-j+1) \leq (n+2)(4\epsilon - 1)^{-1}.$$

If now n is odd, rewrite (2.6) in the form

$$0 \leq \left(j - \frac{n+1}{2}\right)^2 + \frac{n+2}{4\epsilon - 1} - \frac{(n+1)^2}{4}$$

and, retracing the steps, this yields (a).

On the other hand if n is even, rewrite (2.6) as

$$0 \leq \left(j - \frac{n}{2}\right) \left(j - \frac{n+2}{2}\right) + (n+2) \left[\frac{1}{4\epsilon - 1} - \frac{n}{4} \right]$$

from which (b) is deduced.

B. (a) If $r_{j+1} \leq r_{\bar{j}+1}$, then $r_j \leq r_{\bar{j}+1}$, $j=1, 2 \dots n-1$.

(b) If, for n odd, $\epsilon \leq (n+3)^2/4(n+1)^2$ or for n even, $\epsilon \leq (n+4)/4n$ then $r_j \geq r_j^+$ implies $r_{j+1} \geq r_{\bar{j}}$, $j=1, 2 \dots n-1$ (equality for n odd only if $j=(n+1)/2$ and $\epsilon=(n+3)^2/4(n+1)^2$; equality for n even only if $j=n/2$ or $(n+2)/2$, and $\epsilon=(n+4)/4n$).

PROOF: Noting from (2.4) that

$$(2.7) \quad \bar{r}_{j+1} \cdot r_{j+1}^+ = \frac{\epsilon^j(j+1)}{(n-j)(n-j+1)}, \quad i = 1, 2 \cdots n-1,$$

we see from (2.2) and the hypothesis of (a) that

$$\begin{aligned} r_j &\leq \epsilon^j(n-j+1)^{-1} \left[1 - \frac{n-j}{j+1} \bar{r}_{j+1} \right]^{-1} \\ &= \epsilon^j(n-j+1)^{-1} \left[\frac{1}{2} \left(1 + \left(1 - \frac{4\epsilon^j(n-j)}{(j+1)(n-j+1)} \right)^{1/2} \right) \right]^{-1} \\ &= \frac{j+1}{2(n-j)} \left[1 - \left(1 - \frac{4\epsilon^j(n-j)}{(j+1)(n-j+1)} \right)^{1/2} \right] = \bar{r}_{j+1}^-. \end{aligned}$$

To prove (b), we note from A that $\epsilon^j/(n-j+1) \cdot 1/\bar{r}_{j+1}^- > \epsilon^j/(n-j+1) \cdot 1/r_{j+1}^+$, or equivalently

$$\begin{aligned} 1 - \frac{\epsilon^j}{(n-j+1)r_{j+1}^+} &> 1 - \frac{1}{2} \left[1 + \left(1 - \frac{4\epsilon^j(n-j)}{(j+1)(n-j+1)} \right)^{1/2} \right] \\ &= \frac{n-j}{j+1} \bar{r}_{j+1}^-. \end{aligned}$$

Hence, the hypothesis of (b) and (2.2) imply

$$\frac{n-j}{j+1} r_{j+1} \geq 1 - \frac{\epsilon^j}{(n-j+1)r_{j+1}^+} > \frac{n-j}{j+1} \bar{r}_{j+1}^-$$

which yields the conclusion of (b).

In returning to the problem as a whole, we discard the possibility that $\bar{r}_j^-(\epsilon)$ and $r_{j+1}^+(\epsilon)$ are complex valued since this occurs only for values of ϵ larger than those for which we establish solutions. Also, we must separate the case of odd n from that of even n . Suppose first that $n=2k+1$ where k is a non-negative integer. We shall show that $\epsilon_n' = (n+3)^2/4(n+1)^2$ is the smallest value of ϵ for which there exist real numbers $r_1, r_2 \cdots r_n$ satisfying (2.2), (2.2)' and (2.3). As a consequence, no real solution is possible for $\epsilon < \epsilon_n'$. We shall also find a (unique) real solution for $\epsilon = \epsilon_n'$. To this end, we distinguish, via (2.5), three mutually exclusive and exhaustive possibilities:

- (i) $r_{k+1} \geq r_{k+1}^+(\epsilon)$, $r_{k+2} \leq r_{k+2}^-(\epsilon)$,
- (ii) $r_{k+1} \leq r_{k+1}^-(\epsilon)$,
- (iii) $r_{k+2} \geq r_{k+2}^+(\epsilon)$.

If (i) holds, (2.2) and (i) imply

$$(2.8) \quad r_{k+2}^- \geq r_{k+2} \geq \frac{k+2}{k} \left[1 - \frac{\epsilon}{r_{k+1}^+} \right].$$

Substituting (2.7) for $j=k$ into (2.8) and recalling the definitions of r_{k+1}^+, r_{k+2}^- , we find that

$$(2.9) \quad \begin{aligned} & \frac{k+2}{2k} [1 - (1 - 4\epsilon k(k+2)^{-1})^{1/2}] \\ & \geq r_{k+2} \geq \frac{k+2}{k} \left[1 - \frac{k+2}{2k} (1 - (1 - 4\epsilon k(k+2)^{-1})^{1/2}) \right] \end{aligned}$$

which is valid only if $\epsilon \geq (k+2)^2/4(k+1)^2 = (n+3)^2/4(n+1)^2 = \epsilon_n'$. We now show that a real solution does exist for $\epsilon = \epsilon_n'$. When $\epsilon = \epsilon_n'$, (2.9) yields $r_{k+2} = (k+2)/2(k+1)$. Substituting this value of r_{k+2} and $\epsilon = \epsilon_n'$ back in (2.2), we find $r_{k+1} \leq (k+2)/2(k+1)$ which combined with (i) $r_{k+1} \geq r_{k+1}^+(\epsilon_n') = (k+2)/2(k+1)$ implies

$$r_{k+1} = (k+2)/2(k+1) = r_{k+2}.$$

But then, as remarked earlier, $b_j = (k+2)/2(k+1) = (n+3)/2(n+1) = \bar{b}_n'$ (say), $j = 1, 2, \dots, n$ implying $r_j = (n+3)/2(n+1)$, $j = 1, 2, \dots, n$ and these values of r_j are quite compatible with (2.2), (2.2)' and (2.3). Consequently,² $(\epsilon_n', \bar{b}_n')$ is a real solution and among all real solutions conforming to (i), it is the one having minimal ϵ .

Since we are concerned only with the possibility of solutions with smaller ϵ , we assume in what follows that $\epsilon \leq \epsilon_n'$. Suppose next that (ii) $r_{k+1} \leq r_{k+1}^-$ prevails. Part (a) of B yields $r_k \leq r_{k+1}^-$. But then necessarily $r_k \leq r_k^-$ since the only alternative permitted by (2.5), namely $r_k \geq r_k^+$ entails (by proposition A) $r_k \geq r_k^+ > r_{k+1}^-$, a contradiction. Repeating this chain of argument, we eventually deduce $r_2 \leq r_2^-$. Applying (a) of B once more, we find $r_1 \leq r_2^- = [1/n - 1] [1 - (1 - 2\epsilon_n' (n-1)n^{-1})^{1/2}]$ which contradicts (2.2)' for $n \geq 3$.

Finally, the contingency (iii) $r_{k+2} \geq r_{k+2}^+$ implies by (b) of B that $r_{k+3} > r_{k+3}^-$. But then (2.5) insures that $r_{k+3} \geq r_{k+3}^+$. Continuing in this fashion, we find that $r_n \geq r_n^+ > n\epsilon$ (for $\epsilon \leq \epsilon_n'$) contradicting (2.2)' for $n > 1$.

When $n=1$, it is trivial to verify that $\epsilon = \epsilon_1' = 1, b = \bar{b}_1' = 1$ is the unique minimal real solution.

Thus, when $n = 2k + 1, P_{n+2}(x) = (x^2 - x + \epsilon) \prod_{i=1}^n (x + b_i) \in \mathcal{P}^+$ only if $\epsilon \geq \epsilon_n'$. Further, when $\epsilon = \epsilon_n'$ there is only one possible choice of b , namely, $b_1 = b_2 = \dots = b_n = (\epsilon_n')^{1/2}$. These values cause the central terms of $P_{n+2}(x)$ of degree $k+1$ and $k+2$ to vanish.

² It is to be understood wherever a real solution is concerned that (ϵ, z) abbreviates the $(n+1)$ -tuple $(\epsilon, z, z, \dots, z)$.

We turn now to the case $n = 2k$, k a positive integer, and distinguish four mutually exclusive and exhaustive possibilities:

- (i) $r_k \geq r_k^+(\epsilon)$, $r_{k+1} \leq r_{k+1}^-(\epsilon)$;
- (ii) $r_{k+1} \geq r_{k+1}^+(\epsilon)$, $r_{k+2} \leq r_{k+2}^-(\epsilon)$;
- (iii) $r_k \leq r_k^-(\epsilon)$,
- (iv) $r_{k+2} \geq r_{k+2}^+(\epsilon)$.

Under case (i), employing (2.2) and then (2.7) for $j = k$, we have

$$\begin{aligned}
 (2.10) \quad \bar{r}_{k+1}(\epsilon) \geq r_{k+1} &\geq \frac{k+1}{k} \left[1 - \frac{\epsilon k}{(k+1)r_k^+} \right] \\
 &= \frac{k+1}{k} \left[1 - \frac{k+2}{k-1} \bar{r}_k \right]
 \end{aligned}$$

which is valid only when $\epsilon \geq (k+2)/4k = (n+4)/4n = \epsilon_n''$ (say). We show next that a solution exists for $\epsilon = \epsilon_n''$.

For this determination of ϵ , (2.10) implies that $r_{k+1} = 1/2$. Substituting these values back in (2.2) we find $r_k \leq 1/2$, which, combined with (i) $r_k \geq r_k^+(\epsilon_n'') = 1/2$, yields the result $r_k = 1/2 = r_{k+1}$. But then $b_j \equiv 1/2$ which, in turn, forces $r_j \equiv 1/2$ and it is readily checked that these latter values satisfy (2.2) and (2.2)'. Thus,² $(\epsilon_n'', 1/2)$ is a real solution and no other real solution compatible with (i) permits a smaller choice of ϵ .

In case (ii) $r_{k+1} \geq r_{k+1}^+(\epsilon)$, $r_{k+2} \leq r_{k+2}^-(\epsilon)$, analogous reasoning again yields $\epsilon \geq \epsilon_n''$ but now $r_{k+1} = r_{k+2} = (k+2)/2k = (n+4)/2n = 2\epsilon_n''$ and it is easily verified that $(\epsilon_n'', 2\epsilon_n'')$ also constitutes a real solution.²

The remaining cases (iii) $r_k \leq r_k^-(\epsilon)$ and (iv) $r_{k+2} \geq r_{k+2}^+(\epsilon)$ may be disposed of, as in the case n odd, by a systematic use of propositions A and B.

Thus, when $n = 2k$, $P_{n+2}(x) = (x^2 - x + \epsilon) \prod_{i=1}^n (x + b_i) \in \mathcal{O}^+$ only if $\epsilon \geq \epsilon_n''$ and when $\epsilon = \epsilon_n''$, there are two real solutions obtained by choosing $b_j \equiv 2\epsilon_n''$ or $b_j \equiv 1/2$. In the former case the coefficients of the terms of degree k and $k+1$ in $P_{n+2}(x)$ are zero while in the latter case the terms x^k and x^{k-1} vanish. Note that $1/2 < 2\epsilon_n'' \leq 1$ and that $2\epsilon_n'' \searrow 1/2$. Similarly, $1/2 < \bar{b}_n' = (\epsilon_n')^{1/2} \leq 1$ and $\bar{b}_n' \searrow 1/2$.

Define

$$\begin{aligned}
 \epsilon_n &= \epsilon_n', \quad n \text{ odd} & \bar{b}_n &= \bar{b}_n = (\epsilon_n')^{1/2}, \quad n \text{ odd,} \\
 &= \epsilon_n'', \quad n \text{ even} & \bar{b}_n &= 2\epsilon_n'', \quad n \text{ even,} \\
 & & \bar{b}_n &= 1/2, \quad n \text{ even.}
 \end{aligned}$$

Since $\epsilon_n' > \epsilon_{n+1}''$ and $\epsilon_n'' > \epsilon_{n+1}'$, the sequence $\{\epsilon_n\}$ is monotone decreasing (to $1/4$). Hence, the step function (take $\epsilon_0 = +\infty$)

$$N_1(\epsilon) = j, \quad \epsilon_j \leq \epsilon < \epsilon_{j-1}, \quad j = 1, 2 \dots$$

gives the answer to the question initially posed.

Clearly, if $\epsilon' > \epsilon$ and $P_{n+2}(x; \epsilon) = (x^2 - x + \epsilon) \prod_{i=1}^n (x + b_i) \in \mathcal{O}^+$, then $P_{n+2}(x; \epsilon') = P_{n+2}(x; \epsilon) + (\epsilon' - \epsilon) \prod_{i=1}^n (x + b_i) \in \mathcal{O}^+$. Consequently, $(x^2 - x + \epsilon)(x + \bar{b}_n)^n \in \mathcal{O}^+$ for all $\epsilon \geq \epsilon_n$ and likewise $(x^2 - x + \epsilon)(x + \bar{b}_n)^n \in \mathcal{O}^+$, $\epsilon \geq \epsilon_n$. Thus, the answer to the second question raised is affirmative. The real numbers $b_1, b_2 \cdots b_n$ may always be selected equal without increasing n and indeed when $\epsilon = \epsilon_n$, they must be chosen equal if n is to be kept minimal. Expressed otherwise, $N_1(\epsilon) = N_2(\epsilon)$.

3. Remarks. In this section, we give a more precise sufficient condition than that contained in Theorem 1 of [2] which states that a necessary and sufficient condition that there exist $P_m(x) = \sum_{j=0}^m a_j x^j = (x^2 - x + \epsilon) P_{m-2}(x) \in \mathcal{O}^+$ is that $m \geq M(\epsilon)$ where $M(\epsilon)$ is equal to the smallest integer j for which $j \arccos(2(\epsilon)^{1/2})^{-1} \geq \pi$. Here, $m - 2$ and $M(\epsilon) - 2$ correspond respectively to the n and $N_1(\epsilon)$ of §1. We employ the notation and results of [2].

Analogous to (2.1) (of §2 or of [2]), we have (take $P_{m-2}(x) = \sum_{j=0}^{m-2} c_j x^j$)

$$(3.1) \quad c_{j-2} - c_{j-1} + \epsilon c_j = a_j \geq 0, \quad j = 0, 1 \cdots m$$

as a necessary and sufficient condition that $P_m(x) \in \mathcal{O}^+$. Here, $c_{-2} = c_{-1} = c_{m-1} = c_m = 1 - c_{m-2} = 0$.

Multiply (3.1) by $\bar{g}_{j-1} = \bar{g}_{j-1}(\epsilon)$ (see [2] for definition) and sum from $j = 2$ to $j = i$ obtaining (via (1.6) of [2] and $\bar{g}_1 = \bar{g}_2 = 1$)

$$(3.2) \quad c_0 - (\bar{g}_{i-1} - \epsilon \bar{g}_{i-2}) + \epsilon \bar{g}_{i-1} c_i = \sum_{j=2}^i a_j \bar{g}_{j+1}.$$

Hence, from (1.3) of [2],

$$(3.3) \quad c_i \geq \frac{\bar{g}_i(\epsilon) c_{i-1} - c_0}{\epsilon \bar{g}_{i-1}(\epsilon)} \quad \text{for } i < M(\epsilon) - 1.$$

In analogous fashion, multiply (3.1) by $\epsilon^{-(m-2-j)} \bar{g}_{m-1-j}$ and sum from $j = m - 2$ to $j = m - i$ to obtain

$$(3.4) \quad \begin{aligned} \epsilon - \epsilon^{-(i-2)} [\bar{g}_{i-1} - \epsilon \bar{g}_{i-2}] c_{m-i-1} + \epsilon^{-(i-2)} \bar{g}_{i-1} c_{m-i-2} \\ = \sum_{r=2}^i \epsilon^{-(r-2)} \bar{g}_{r-1} a_{m-r}, \end{aligned}$$

implying

$$(3.5) \quad c_j \geq \frac{\bar{g}_{m-j-2} c_{j+1} - \epsilon^{m-j-3}}{\bar{g}_{m-j-3}} \quad \text{for } m - j < M(\epsilon) - 1.$$

Now, the trigonometric representation (see (1.3) of [2]) of $g_j(\delta)$ readily yields the identity $g_j^2(\delta) - (1 + \delta)^{j-1} = g_{j-1}(\delta)g_{j+1}(\delta)$ or equivalently

$$(3.6) \quad \bar{g}_j^2(\epsilon) - \epsilon^{j-1} = \bar{g}_{j-1}(\epsilon)\bar{g}_{j+1}(\epsilon), \quad j \geq 2.$$

Then, utilizing (3.6), an induction on (3.3) gives

$$(3.7) \quad c_j \geq \epsilon^{-j}\bar{g}_{j+1}(\epsilon)c_0, \quad 1 \leq j < M(\epsilon) - 1.$$

Similarly, induction on (3.5) leads to

$$(3.8) \quad c_{m-j} \geq \bar{g}_{j-1}(\epsilon), \quad m - j < M(\epsilon) - 1.$$

Thus, (3.7) and (3.8) furnish simple necessary conditions for the coefficients c_j in terms of the known polynomials $\bar{g}_j(\epsilon)$.

Finally, we note that choosing c_j equal to the lower bound of (3.8) gives

$$\begin{aligned} P_m(x) &= (x^2 - x + \epsilon) \sum_{i=0}^{m-2} \bar{g}_{m-i-1}(\epsilon)x^i = x^m + (\epsilon\bar{g}_{m-2} - \bar{g}_{m-1})x + \epsilon\bar{g}_{m-1}(\epsilon) \\ &= x^m - \bar{g}_m(\epsilon)x + \epsilon\bar{g}_{m-1}(\epsilon). \end{aligned}$$

But for $m = M(\epsilon)$ (as well as for a range of larger values), $g_m(\epsilon) \leq 0$ whence $P_m(x) \in \mathcal{P}^+$, $m = M(\epsilon)$.

A trinomial of this form was constructed in the sufficiency proof of Theorem 1 of [2] but the preceding gives the nonvanishing coefficients as explicit functions of ϵ and at the same time specifies the coefficients of $(x^2 - x + \epsilon)^{-1}P_m(x)$.

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