

A COMPLETE ORTHONORMAL SYSTEM OF HOMOGENEOUS POLYNOMIALS ON MATRIX SPACES OF ORDER 2

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1. **Introduction.** Let

$$z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$$

be a matrix of complex numbers, z^* its conjugate transpose and I the identity matrix. Let D be the domain $E[z | I - zz^* > 0]$. The set $B = E[z | zz^* = I]$ forms a proper part of the boundary of D . Hua [3; 4] has shown by means of representation theory that there exists an orthonormal system (ONS) of homogeneous polynomials in D with respect to the inner product $\int_D f \bar{g} dV_D$. In this paper we exhibit such a system $\{\phi_j\}$ explicitly on D in a special case, only with respect to the inner product

$$(1) \quad (\phi_j, \phi_k) = \int_B \phi_j \bar{\phi}_k dV = \delta_{jk}$$

(dV Euclidean volume element on B). The functions turn out to be certain hypergeometric functions multiplied by powers of z_j .

2. Calculation of the inner product.

LEMMA. *A parametrization of the set $zz^* = I$ is*

$$(1) \quad \begin{aligned} z_1 &= r e^{i\theta_1}, \\ z_2 &= [1 - r^2]^{1/2} e^{i\theta_2}, \\ z_3 &= [1 - r^2]^{1/2} e^{i\theta_3}, \\ z_4 &= -r e^{i(-\theta_1 + \theta_2 + \theta_3)}, \end{aligned}$$

($0 \leq r \leq 1$, $0 \leq \theta_j \leq 2\pi$, $j = 1, 2, 3$).

PROOF. If we set $z_j = r_j e^{i\theta_j}$, then $zz^* = I$ if and only if $r_1^2 + r_2^2 = 1$, $r_3^2 + r_4^2 = 1$, $r_1 r_3 + r_2 r_4 \cos \alpha = 0$, $r_2 r_4 \sin \alpha = 0$ ($\alpha = -\theta_1 + \theta_2 + \theta_3 - \theta_4$). The last equation implies $r_2 = 0$, $r_4 = 0$ or $\alpha = n\pi$ ($n = 0, \pm 1, \dots$). Now $r_2 = 0$ implies $r_1 = 1$, $r_3 = 0$, $r_4 = 1$. However the set

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$$E[z \mid z_1\bar{z}_1 = 1, z_4\bar{z}_4 = 1, z_2 = z_3 = 0]$$

is the Cartesian product of 2 circles lying in the planes $\text{Re } z_2 = \text{Re } z_3 = \text{Im } z_2 = \text{Im } z_3 = 0$ and hence is a two-dimensional subset of the four-dimensional set B and as such may be disregarded in the integration over B . Similarly if $r_4 = 0$. Thus we may take $\alpha = \pm\pi$. Hence $r_1r_3 = r_2r_4$, also $r_2 = r_3$ and $r_1 = r_4$. Setting $r_1 = r$, $\theta_4 = \pm\pi - \theta_1 + \theta_2 + \theta_3$, (1) follows.

Now

$$dV = idz_1dz_2dz_3dz_4/\det^2 z = iJdrd\theta_1d\theta_2d\theta_3/\det^2 z$$

[7], J being the Jacobian of z with respect to r and θ . But

$$J = -2ire^{2i(\theta_2+\theta_3)},$$

and

$$\det z = -e^{i(\theta_2+\theta_3)},$$

so that

$$(2) \quad (f, g) = 2 \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} f\bar{g}rdrd\theta_1d\theta_2d\theta_3.$$

In order to construct the ONS we must evaluate integrals of the type

$$a_{jk} = (P_j, P_k),$$

where

$$(3) \quad P_j = z_1^p z_2^q (z_1 z_4)^j (z_2 z_3)^{r-j} \quad (j = 0, 1, \dots, r)$$

(cf. Lemma 3.2). By (1) and (2)

$$a_{jk} = (-1)^{j+k} 16\pi^3 \int_0^1 r^{1+2p+2j+2k} (1-r^2)^{q+2r-j-k} dr.$$

The substitution $t=r^2$ transforms a_{jk} into Dirichlet's integral which has the value [8]

$$(4) \quad a_{jk} = \frac{(-1)^{j+k} 8\pi^3 (p+j+k)!(Q-j-k)!}{(p+Q+1)!} \quad (Q = q + 2r).$$

3. System of homogeneous polynomials. Let

$$(1) \quad P(z) = \prod_{j=1}^4 z_j^{p_j} \quad Q(z) = \prod_{j=1}^4 z_j^{q_j},$$

p_j and q_j being non-negative integers. By means of (2.1) and (2.2) it is seen that $(P, Q) \neq 0$ if and only if

$$(2) \quad \begin{aligned} p_1 - q_1 &= a, \\ p_2 - q_2 &= -a, \\ p_3 - q_3 &= -a, \\ p_4 - q_4 &= a, \end{aligned}$$

where a is an integer.² Thus setting $q_1 = p, p_2 = q, p_3 = r, q_4 = s$, we get

LEMMA 1. *Let P and Q be powers in z_j . The inner product $(P, Q) \neq 0$ if and only if*

$$(3) \quad \begin{aligned} P(z) &= z_1^p z_2^q z_3^r z_4^s (z_1 z_4)^a, \\ Q(z) &= z_1^p z_2^q z_3^r z_4^s (z_2 z_3)^a, \end{aligned}$$

$$(p, q, r, s = 0, 1, 2, \dots; a \geq \sup(-p, -q, -r, -s)).$$

LEMMA 2. *Let*

$$\begin{aligned} (i) \quad P_{pqr}^{(1j)}(z) &= P_{1j}(z) = z_1^p z_2^q (z_1 z_4)^j (z_2 z_3)^{r-j}, \\ (ii) \quad P_{pqr}^{(2j)}(z) &= P_{2j}(z) = z_1^p z_3^q (z_1 z_4)^j (z_2 z_3)^{r-j}, \\ (iii) \quad P_{pqr}^{(3j)}(z) &= P_{3j}(z) = z_4^p z_2^q (z_1 z_4)^j (z_2 z_3)^{r-j}, \\ (iv) \quad P_{pqr}^{(4j)}(z) &= P_{4j}(z) = z_4^p z_3^q (z_1 z_4)^j (z_2 z_3)^{r-j}, \end{aligned}$$

$(j = 0, 1, \dots, r; p, q, r = 0, 1, 2, \dots)$. Then $(P_{pqr}^{(ij)}, P_{p'q'r'}^{(i'j')}) \neq 0$, if and only if $i = i', p = p', q = q'$ and $r = r'$.

PROOF. Let $P(z)$ be given by (1). Suppose $p_4 = \min(p_1, p_4)$, $p_3 = \min(p_2, p_3)$. Define $j = p_4, p = p_1 - p_4, r = p_3 + p_4, q = p_2 - p_3$. This expresses P in the form (2.3). A power $Q = z_1^{q_1} z_2^{q_2} z_3^{q_3} z_4^{q_4}$ is nonorthogonal to P if and only if, in case $q_4 \leq p_4$,

$$q_1 = p + j - a, q_2 = q + r - (j - a), q_3 = r - (j - a), q_4 = j - a$$

$(a = 0, 1, \dots, j)$, or, in case $q_4 \geq p_4$,

$$q_1 = p + j + a, q_2 = q + r - j - a, q_3 = r - j - a, q_4 = j + a$$

$(a = 0, 1, \dots, r - j)$, that is, in case

$$Q(z) = z_1^p z_2^q (z_1 z_4)^k (z_2 z_3)^{r-k} \quad (k = 0, 1, \dots, r).$$

This proves (i) of Lemma 2 and similarly for the other parts.

² Conditions (2) may be generalized to n by n matrices if integration is over D , thus giving a procedure for subdividing the powers of a fixed degree into non-orthogonal subsets.

4. ONS of homogeneous polynomials.

THEOREM. An ONS of homogeneous polynomials on D orthonormalized with respect to the inner product (1.1) is

$$(1) \quad \phi_{pqr}^{(ij)}(z) = N_{pqr}^{(j)} P_{ij}(z) F(-j, -j - p; Q + 2 - 2j; Z)$$

($i=1, \dots, 4; j=0, \dots, r; p, q, r=0, 1, 2, \dots; q \neq 0$ for $i=2, p \neq 0$ for $i=3, pq \neq 0$ for $i=4$) where F is the hypergeometric function and $Z = z_2 z_3 / z_1 z_4$. The normalizing factor is

$$(2) \quad N^{(j)} = N_{pqr}^{(j)} = \left[\frac{j+1}{8\pi^3} \binom{p+Q+1-j}{p+j} \binom{Q+1-j}{j+1} \right]^{1/2}.$$

PROOF. The hypergeometric function has the finite expansion

$$(3) \quad (a) \quad F(-j, -j - p; Q + 2 - 2j; Z) = \sum_{\nu=0}^j C_{j,\nu} Z^\nu,$$

$$(b) \quad C_{j\nu} = \frac{\binom{j}{\nu} \binom{p+j}{j-\nu}}{\binom{Q+1-j-\nu}{j-\nu}}.$$

Orthogonality. From (3.2) and Lemma 2 ($\phi_{pqr}^{(jl)}, \phi_{stu}^{(lk)} = 0$ if $l \neq i$). Similarly ($\phi_{pqr}^{(jl)}, \phi_{stu}^{(lk)} = 0$ unless $p=s, q=t$ and $r=u$). It only remains to show that

$$(4) \quad I_{jk} = (\phi^{(ij)}, \phi^{(ik)}) = 0 \quad \text{for } j \neq k,$$

($\phi^{(ii)} = \phi_{pqr}^{(ij)}$). Without loss of generality we may assume $k < j$ and $i=1$. Then

$$I_{jk} = N \sum_{\nu=0}^j \sum_{\mu=0}^k C_{j,j-\nu} C_{k,k-\mu} a_{j-\nu,k-\mu}$$

($N = N_{pqr}^{(j)} N_{pqr}^{(k)}$), or replacing ν by $j-\nu$ and μ by $k-\mu$,

$$(5) \quad I_{jk} = N \sum_{\mu=0}^k C_{k\mu} I'_{j\mu},$$

where

$$(6) \quad I'_{j\mu} = \sum_{\nu=0}^j C_{j\nu} a_{\nu\mu}.$$

We prove (4) by induction on k for fixed j . Specifically we show that

$I_{j0} = NI'_{j0} = 0$ and that $I'_{j\sigma} = 0$ for $\sigma = 0, 1, \dots, k-1$ implies $I'_{jk} = 0$. Hence $I_{jk} = 0$.

(i) For $k = 0$,

$$I_{j0} = NC_{00} \sum_{\nu=0}^j C_{j\nu} a_{\nu 0} = A_j S_{j0},$$

where

$$(7) \quad A_j = (-1)^j 8\pi^3 N \frac{(p+j)!(Q+1-2j)!}{(Q+p+1)!}$$

and

$$(8) \quad S_{j0} = \sum_{\nu=0}^j (-1)^{j-\nu} \binom{j}{\nu} \frac{(Q-\nu)!}{(Q+1-j-\nu)!}.$$

By using the binomial theorem and then differentiating the expression

$$(9) \quad D_{j-1,k}(x, y) = \frac{\partial^{j-1}}{\partial x^k \partial y^{j-1-k}} x^{p+k} y^{Q-j-k} (x-y)^j,$$

(defined for $k < j$), we find that

$$S_{j0} = D_{j-1,0}(1, 1).$$

Upon differentiating (9) we see that $D_{j-1,k}(x, y)$ equals $x-y$ times a polynomial in x and y . Hence $D_{j-1,k}(1, 1) = 0$ for all $k < j$. Thus S_{j0} and I_{j0} are both 0.

(ii) Let k be such that $0 \leq k \leq j-1$. Assuming by induction hypothesis that $I'_{j\sigma} = 0$ for $\sigma = 0, 1, \dots, k-1$, we find by using the binomial theorem on $D_{j-1,k}(x, y)$ and then differentiating that I_{jk} is a multiple of $D_{j-1,k}(1, 1)$. Thus I'_{jk} and I_{jk} are 0 and the set (1) is orthogonal.

Normality. We have to show that $I_{jj} = 1$. By (5) and the fact that $I'_{j\mu} = 0$ for $\mu = 0, 1, \dots, j-1$,

$$I_{jj} = NC_{jj} I'_{jj} = N \sum_{\nu=0}^j C_{j\nu} a_{\nu j},$$

which by (2.4), (3b) and (7) has the value

$$A_j \sum_{\nu=0}^j (-1)^\nu \binom{j}{\nu} \frac{(p+\nu+1) \cdots (p+\nu+j)}{Q+1-j-\nu}.$$

Now the expression

$$T_j = (-1)^j A_j \frac{\partial^j}{\partial x^j} \left\{ x^{p+i} \int_0^1 y^{Q-2i}(x-y)^i dy \right\}$$

can be shown to equal I_{jj} when $x=1$ by first using the binomial theorem and then performing the integration and differentiation. On the other hand integrating T_j iteratively, then differentiating and setting $x=1$, we find that only one term is different from 0 and its value is

$$(-1)^j A_j \frac{j!(Q+p+1)!(Q-2j)!}{(Q+p+1-j)!(Q+1-j)!}$$

which equals 1. Thus (1) is an ONS.

By analogous methods but much more complicated computations it may be shown that the system

$$(10) \quad \psi_{pqr}^{(ij)} = M_{pqr}^{(j)} P_{ij}(z) F(-j, -j-p; Q+4-2j; Z)$$

($i=1, \dots, 4; j=0, \dots, r; p, q, r=0, 1, 2, \dots; q \neq 0$ for $i=2, p \neq 0$ for $i=3, pq \neq 0$ for $i=4$) where

$$(11) \quad M_{pqr}^{(j)} = \left[\frac{4(j+1)}{\pi^4} \binom{r-j+2}{2} \binom{q+r-j+2}{2} \binom{Q+p+3-j}{p+j} \binom{Q+3-j}{j+1} \right]^{1/2}$$

forms an ONS on D with respect to the inner product

$$(12) \quad \int_D f \bar{g} dV_D.$$

(Cf. [5], where some of the calculations are carried out and the general procedure is explained.)

5. Completeness. Hua [4] has proved that the set of powers (3.1) is complete on D for functions of class $L^{2,3}$ that is, if f is an analytic function with finite norm $[\int_D |f|^2 dV_D]^{1/2}$, on D , and

$$\int_D f \bar{P} dV_D = 0, \quad (p_j = 0, 1, 2, \dots; j = 1, \dots, 4),$$

then $f=0$ on D . Now each orthogonal set of (4.1) ($j=0, \dots, r$) is

³ Class L^2 was first introduced by Bergman (cf. [1]).

formed from $r + 1$ distinct powers of the type described in Lemma 2. Thus

$$\phi_{ij} = \sum_{r=0}^j C_{jr} P_{ir} \quad (j = 0, \dots, r; C_{jj} \neq 0).$$

Consequently

$$P_{ij} = \sum_{r=0}^j d_{jr} \phi_{ir}.$$

Furthermore the proofs of Lemmas 1 and 2 show that every power P belongs to one and only one set $\{P_{ij}\}$ ($i=1, \dots, 4$). Consequently $\{\phi\}$ is complete with respect to functions of class L^2 on D . Similarly for the ONS (4.10).

6. Expansion theorems. By a result due to Bergman [1] any function f of class L^2 on D has an orthogonal development

$$(1) \quad S = \sum_{n=0}^{\infty} a_n \psi_n, \quad a_n = \int_D f \bar{\psi}_n dV,$$

and (1) converges absolutely and continuously to f in D ($\{\psi_n\}$ is any convenient ordering of the ONS (4.10)). Also if $\{a_n\}$ is an arbitrary sequence of constants such that $\sum |a_n|^2$ converges, then the series S converges absolutely and continuously in D . In our case this gives a theorem on the convergence of a series of hypergeometric functions.

Also (4.10) gives an expansion for the Bergman kernel function [6]

$$K(z, t) = 12/[\pi^4 \det^4 (I - zt^*)],$$

namely,

$$K(z, t) = \sum_{n=0}^{\infty} \psi_n(z) \bar{\psi}_n(t^*),$$

in terms of hypergeometric functions for all z, t in D . Analogous results hold for the Szegö kernel,

$$1/[8\pi^3 \det^2 (I - zt^*)]$$

[2], related to the ONS (4.1).

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