A COMPLETE ORTHONORMAL SYSTEM OF HOMOGENEOUS POLYNOMIALS ON MATRIX SPACES OF ORDER 2

JOSEPHINE MITCHELL[1]

1. Introduction. Let

$$z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$$

be a matrix of complex numbers, z^* its conjugate transpose and I the identity matrix. Let D be the domain $E[z|I-zz^*>0]$. The set $B=E[z|zz^*=I]$ forms a proper part of the boundary of D. Hua [3; 4] has shown by means of representation theory that there exists an orthonormal system (ONS) of homogeneous polynomials in D with respect to the inner product $\int_D f\bar{g}dV_D$. In this paper we exhibit such a system $\{\phi_j\}$ explicitly on D in a special case, only with respect to the inner product

$$(1) \qquad (\phi_j, \phi_k) = \int_B \phi_j \bar{\phi}_k dV = \delta_{jk}$$

(dV Euclidean volume element on B). The functions turn out to be certain hypergeometric functions multiplied by powers of z_j.

2. Calculation of the inner product.

LEMMA. *A parametrization of the set $zz^*=I$ is*

$$(1) \qquad \begin{aligned} z_1 &= re^{i\theta_1}, \\ z_2 &= [1-r^2]^{1/2}e^{i\theta_2}, \\ z_3 &= [1-r^2]^{1/2}e^{i\theta_3}, \\ z_4 &= -re^{i(-\theta_1+\theta_2+\theta_3)}, \end{aligned}$$

$(0\leqq r\leqq 1,\ 0\leqq\theta_j\leqq 2\pi,\ j=1, 2, 3)$.

PROOF. If we set $z_j=r_je^{i\theta_j}$, then $zz^*=I$ if and only if $r_1^2+r_2^2=1$, $r_3^2+r_4^2=1$, $r_1r_3+r_2r_4\cos\alpha=0$, $r_2r_4\sin\alpha=0$ ($\alpha=-\theta_1+\theta_2+\theta_3-\theta_4$). The last equation implies $r_2=0$, $r_4=0$ or $\alpha=n\pi$ ($n=0, \pm 1, \cdots$). Now $r_2=0$ implies $r_1=1$, $r_3=0$, $r_4=1$. However the set

Presented to the Society, April 20, 1956 under the title *Orthogonal systems in matric spaces*. Preliminary report; received by the editors August 2, 1958 and, in revised form, September 29, 1958.

[1] The research in this paper was done while the author was the holder of an A.A.U.W. National Fellowship in 1954–1955.

$$E[z \mid z_1\bar{z}_1 = 1,\ z_4\bar{z}_4 = 1,\ z_2 = z_3 = 0]$$

is the Cartesian product of 2 circles lying in the planes $\operatorname{Re} z_2 = \operatorname{Re} z_3 = \operatorname{Im} z_2 = \operatorname{Im} z_3 = 0$ and hence is a two-dimensional subset of the four-dimensional set B and as such may be disregarded in the integration over B. Similarly if $r_4 = 0$. Thus we may take $\alpha = \pm\pi$. Hence $r_1 r_3 = r_2 r_4$, also $r_2 = r_3$ and $r_1 = r_4$. Setting $r_1 = r$, $\theta_4 = \pm\pi - \theta_1 + \theta_2 + \theta_3$, (1) follows.

Now

$$dV = i dz_1 dz_2 dz_3 dz_4/\det^2 z = iJ dr d\theta_1 d\theta_2 d\theta_3/\det^2 z$$

[7], J being the Jacobian of z with respect to r and θ. But

$$J = -2ire^{2i(\theta_2+\theta_3)},$$

and

$$\det z = -e^{i(\theta_2+\theta_3)},$$

so that

$$(f, g) = 2\int_0^1 \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} f\bar{g} r dr d\theta_1 d\theta_2 d\theta_3. \tag{2}$$

In order to construct the ONS we must evaluate integrals of the type

$$a_{jk} = (P_j, P_k),$$

where

$$P_j = z_1^p z_2^q (z_1 z_4)^j (z_2 z_3)^{r-j} \qquad (j = 0, 1, \cdots, r) \tag{3}$$

(cf. Lemma 3.2). By (1) and (2)

$$a_{jk} = (-1)^{j+k} 16\pi^3 \int_0^1 r^{1+2p+2j+2k}(1 - r^2)^{q+2r-j-k} dr.$$

The substitution $t = r^2$ transforms a_{jk} into Dirichlet's integral which has the value [8]

$$a_{jk} = \frac{(-1)^{j+k} 8\pi^3 (p + j + k)!(Q - j - k)!}{(p + Q + 1)!} \tag{4}$$

$$(Q = q + 2r).$$

3. System of homogeneous polynomials. Let

$$P(z) = \prod_{j=1}^{4} z_j^{p_j}, \qquad Q(z) = \prod_{j=1}^{4} z_j^{q_j}, \tag{1}$$

p_j and q_j being non-negative integers. By means of (2.1) and (2.2) it is seen that $(P, Q)\neq 0$ if and only if

$$\begin{aligned} p_1 - q_1 &= a, \\ p_2 - q_2 &= -a, \\ p_3 - q_3 &= -a, \\ p_4 - q_4 &= a, \end{aligned} \tag{2}$$

where a is an integer.[2] Thus setting $q_1=p$, $p_2=q$, $p_3=r$, $q_4=s$, we get

LEMMA 1. *Let P and Q be powers in z_j. The inner product $(P, Q)\neq 0$ if and only if*

$$\begin{aligned} P(z) &= z_1^p z_2^q z_3^r z_4^s (z_1 z_4)^a, \\ Q(z) &= z_1^p z_2^q z_3^r z_4^s (z_2 z_3)^a, \end{aligned} \tag{3}$$

$(p, q, r, s=0, 1, 2, \cdots; a\geqq \sup(-p, -q, -r, -s))$.

LEMMA 2. *Let*

$$\begin{aligned} &\text{(i)} && P^{(1j)}_{pqr}(z) = P_{1j}(z) = z_1^p z_2^q (z_1z_4)^j (z_2z_3)^{r-j}, \\ &\text{(ii)} && P^{(2j)}_{pqr}(z) = P_{2j}(z) = z_1^p z_3^q (z_1z_4)^j (z_2z_3)^{r-j}, \\ &\text{(iii)} && P^{(3j)}_{pqr}(z) = P_{3j}(z) = z_4^p z_2^q (z_1z_4)^j (z_2z_3)^{r-j}, \\ &\text{(iv)} && P^{(4j)}_{pqr}(z) = P_{4j}(z) = z_4^p z_3^q (z_1z_4)^j (z_2z_3)^{r-j}, \end{aligned}$$

$(j=0, 1, \cdots, r; p, q, r=0, 1, 2, \cdots)$. *Then $(P^{(ij)}_{pqr}, P^{(i'j')}_{(p'q'r')})\neq 0$, if and only if $i=i'$, $p=p'$, $q=q'$ and $r=r'$.*

PROOF. Let $P(z)$ be given by (1). Suppose $p_4=\min(p_1, p_4)$, $p_3=\min(p_2, p_3)$. Define $j=p_4$, $p=p_1-p_4$, $r=p_3+p_4$, $q=p_2-p_3$. This expresses P in the form (2.3). A power $Q=z_1^{q_1}z_2^{q_2}z_3^{q_3}z_4^{q_4}$ is nonorthogonal to P if and only if, in case $q_4\leqq p_4$,

$$q_1 = p + j - a,\ q_2 = q + r - (j - a),\ q_3 = r - (j - a),\ q_4 = j - a$$

$(a=0, 1, \cdots, j)$, or, in case $q_4\geqq p_4$,

$$q_1 = p + j + a,\ q_2 = q + r - j - a,\ q_3 = r - j - a,\ q_4 = j + a$$

$(a=0, 1, \cdots, r-j)$, that is, in case

$$Q(z) = z_1^p z_2^q (z_1z_4)^k (z_2z_3)^{r-k} \qquad (k = 0, 1, \cdots, r).$$

This proves (i) of Lemma 2 and similarly for the other parts.

[2] Conditions (2) may be generalized to n by n matrices if integration is over D, thus giving a procedure for subdividing the powers of a fixed degree into non-orthogonal subsets.

4. ONS of homogeneous polynomials.

THEOREM. *An ONS of homogeneous polynomials on D orthonormalized with respect to the inner product* (1.1) *is*

$$\phi_{pqr}^{(ij)}(z) = N_{pqr}^{(j)} P_{ij}(z) F(-j, -j-p; Q+2-2j; Z) \tag{1}$$

($i=1, \cdots, 4$; $j=0, \cdots, r$; $p, q, r=0, 1, 2, \cdots$; $q\neq 0$ for $i=2$, $p\neq 0$ for $i=3$, $pq\neq 0$ for $i=4$) *where F is the hypergeometric function and* $Z=z_2z_3/z_1z_4$. *The normalizing factor is*

$$N^{(j)} = N_{pqr}^{(j)} = \left[\frac{j+1}{8\pi^3}\binom{p+Q+1-j}{p+j}\binom{Q+1-j}{j+1}\right]^{1/2}. \tag{2}$$

PROOF. The hypergeometric function has the finite expansion

$$\text{(3)}\qquad \begin{aligned} &\text{(a)}\ F(-j, -j-p; Q+2-2j; Z) = \sum_{\nu=0}^{j} C_{j,j-\nu} Z^{\nu}, \\ &\text{(b)}\ C_{j\nu} = \frac{\binom{j}{\nu}\binom{p+j}{j-\nu}}{\binom{Q+1-j-\nu}{j-\nu}}. \end{aligned}$$

Orthogonality. From (3.2) and Lemma 2 $(\phi_{pqr}^{(ij)}, \phi_{stu}^{(lk)})=0$ if $l\neq i$. Similarly $(\phi_{pqr}^{(ij)}, \phi_{stu}^{(ik)})=0$ unless $p=s$, $q=t$ and $r=u$. It only remains to show that

$$I_{jk} = (\phi^{(ij)}, \phi^{(ik)}) = 0 \qquad \text{for } j\neq k, \tag{4}$$

$(\phi^{(ij)}=\phi_{pqr}^{(ij)})$. Without loss of generality we may assume $k<j$ and $i=1$. Then

$$I_{jk} = N\sum_{\nu=0}^{j}\sum_{\mu=0}^{k} C_{j,j-\nu}C_{k,k-\mu}a_{j-\nu,k-\mu}$$

$(N=N_{pqr}^{(j)}N_{pqr}^{(k)})$, or replacing ν by $j-\nu$ and μ by $k-\mu$,

$$I_{jk} = N\sum_{\mu=0}^{k} C_{k\mu}I'_{j\mu}, \tag{5}$$

where

$$I'_{j\mu} = \sum_{\nu=0}^{j} C_{j\nu}a_{\nu\mu}. \tag{6}$$

We prove (4) by induction on k for fixed j. Specifically we show that

$I_{j0}=NI'_{j0}=0$ and that $I'_{j\sigma}=0$ for $\sigma=0, 1, \cdots, k-1$ implies $I'_{jk}=0$. Hence $I_{jk}=0$.

(i) For $k=0$,

$$I_{j0} = NC_{00}\sum_{\nu=0}^{j} C_{j\nu}a_{\nu 0}$$
$$= A_j S_{j0},$$

where

$$A_j = (-1)^j 8\pi^3 N \frac{(p+j)!(Q+1-2j)!}{(Q+p+1)!} \tag{7}$$

and

$$S_{j0} = \sum_{\nu=0}^{j} (-1)^{j-\nu}\binom{j}{\nu}\frac{(Q-\nu)!}{(Q+1-j-\nu)!}\,. \tag{8}$$

By using the binomial theorem and then differentiating the expression

$$D_{j-1,k}(x, y) = \frac{\partial^{j-1}}{\partial x^k \partial y^{j-1-k}}\, x^{p+k}y^{Q-j-k}(x-y)^j, \tag{9}$$

(defined for $k<j$), we find that

$$S_{j0} = D_{j-1,0}(1, 1).$$

Upon differentiating (9) we see that $D_{j-1,k}(x, y)$ equals $x-y$ times a polynomial in x and y. Hence $D_{j-1,k}(1, 1)=0$ for all $k<j$. Thus S_{j0} and I_{j0} are both 0.

(ii) Let k be such that $0\leqq k\leqq j-1$. Assuming by induction hypothesis that $I'_{j\sigma}=0$ for $\sigma=0, 1, \cdots, k-1$, we find by using the binomial theorem on $D_{j-1,k}(x, y)$ and then differentiating that I_{jk} is a multiple of $D_{j-1,k}(1, 1)$. Thus I'_{jk} and I_{jk} are 0 and the set (1) is orthogonal.

Normality. We have to show that $I_{jj}=1$. By (5) and the fact that $I'_{j\mu}=0$ for $\mu=0, 1, \cdots, j-1$,

$$I_{jj} = NC_{jj}I'_{jj} = N\sum_{\nu=0}^{j} C_{j\nu}a_{\nu j},$$

which by (2.4), (3b) and (7) has the value

$$A_j\sum_{\nu=0}^{j}(-1)^\nu\binom{j}{\nu}\frac{(p+\nu+1)\cdots(p+\nu+j)}{Q+1-j-\nu}\,.$$

Now the expression

$$T_j = (-1)^j A_j \frac{\partial^j}{\partial x^j}\left\{ x^{p+j} \int_0^1 y^{Q-2j}(x-y)^j dy \right\}$$

can be shown to equal I_{jj} when $x=1$ by first using the binomial theorem and then performing the integration and differentiation. On the other hand integrating T_j iteratively, then differentiating and setting $x=1$, we find that only one term is different from 0 and its value is

$$(-1)^j A_j \frac{j!(Q+p+1)!(Q-2j)!}{(Q+p+1-j)!(Q+1-j)!}$$

which equals 1. Thus (1) is an ONS.

By analogous methods but much more complicated computations it may be shown that the system

$$\psi_{pqr}^{(ij)} = M_{pqr}^{(j)} P_{ij}(z) F(-j, -j-p; Q+4-2j; Z) \tag{10}$$

($i=1, \cdots, 4$; $j=0, \cdots, r$; $p, q, r=0, 1, 2, \cdots$; $q \neq 0$ for $i=2$, $p \neq 0$ for $i=3$, $pq \neq 0$ for $i=4$) where

$$M_{pqr}^{(j)} = \left[\frac{4(j+1)}{\pi^4}\binom{r-j+2}{2}\binom{q+r-j+2}{2} \binom{Q+p+3-j}{p+j}\binom{Q+3-j}{j+1}\right]^{1/2} \tag{11}$$

forms an ONS on D with respect to the inner product

$$\int_D f\bar{g}\,dV_D. \tag{12}$$

(Cf. [5], where some of the calculations are carried out and the general procedure is explained.)

5. **Completeness.** Hua [4] has proved that the set of powers (3.1) is complete on D for functions of class L^2,[3] that is, if f is an analytic function with finite norm $[\int_D |f|^2 dV_D]^{1/2}$, on D, and

$$\int_D f\bar{P}\,dV_D = 0, \qquad (p_j = 0, 1, 2, \cdots; j = 1, \cdots, 4),$$

then $f=0$ on D. Now each orthogonal set of (4.1) ($j=0, \cdots, r$) is

[3] Class L^2 was first introduced by Bergman (cf. [1]).

formed from $r+1$ distinct powers of the type described in Lemma 2. Thus

$$\phi_{ij} = \sum_{\nu=0}^{j} C_{j\nu} P_{i\nu} \qquad (j = 0, \cdots, r; C_{jj} \neq 0).$$

Consequently

$$P_{ij} = \sum_{\nu=0}^{j} d_{j\nu} \phi_{i\nu}.$$

Furthermore the proofs of Lemmas 1 and 2 show that every power P belongs to one and only one set $\{P_{ij}\}$ $(i=1, \cdots, 4)$. Consequently $\{\phi\}$ is complete with respect to functions of class L^2 on D. Similarly for the ONS (4.10).

6. **Expansion theorems.** By a result due to Bergman [**1**] any function f of class L^2 on D has an orthogonal development

$$S = \sum_{n=0}^{\infty} a_n \psi_n, \qquad a_n = \int_D f \bar{\psi}_n dV, \tag{1}$$

and (1) converges absolutely and continuously to f in D($\{\psi_n\}$ is any convenient ordering of the ONS (4.10)). Also if $\{a_n\}$ is an arbitrary sequence of constants such that $\sum |a_n|^2$ converges, then the series S converges absolutely and continuously in D. In our case this gives a theorem on the convergence of a series of hypergeometric functions.

Also (4.10) gives an expansion for the Bergman kernel function [**6**]

$$K(z, t) = 12/[\pi^4 \det^4 (I - zt^*)],$$

namely,

$$K(z, t) = \sum_{n=0}^{\infty} \psi_n(z) \psi_n(t^*),$$

in terms of hypergeometric functions for all z, t in D. Analogous results hold for the Szegö kernel,

$$1/[8\pi^3 \det^2 (I - zt^*)]$$

[**2**], related to the ONS (4.1).

References

1. S. Bergman, *The kernel function and conformal mapping*, Mathematical Surveys, vol. 5, American Mathematical Society, 1950.

2. S. Bochner, *Group invariance of Cauchy's formula in several variables*, Ann. of Math. vol. 45 (1944) pp. 686–707.

3. L. K. Hua, *On the theory of functions of several complex variables.* I. *On a complete orthonormal system in the hyperbolic space of rectangular matrices*, Dokl. Akad. Nauk SSSR (N.S.) vol. 93 (1953) pp. 775–777 [Math. Rev. 15 (1954) p. 617] and J. Chinese Math. Soc. vol. 2 (1953) pp. 288–323 [Math. Rev. 17, 1956, p. 191].

4. ———, *Harmonic analysis of the classical domain in the study of analytic functions of several complex variables*, Mimeographed Lecture Notes, about 1956.

5. J. Mitchell, *An example of a complete orthonormal system and the kernel function in the geometry of matrices*, Proceedings of Second Canadian Mathematical Congress, Vancouver, 1949, pp. 155–163.

6. ———, *The kernel function in the geometry of matrices*, Duke Math. J. vol. 19 (1952) pp. 575–584.

7. ———, *Potential theory in the geometry of matrices*, Trans. Amer. Math. Soc. vol. 79 (1955) pp. 401–422.

8. E. T. Whittaker and G. N. Watson, *A course of modern analysis*, Cambridge, Cambridge University Press, 1948.

PENNSYLVANIA STATE UNIVERSITY