CERTAIN OPERATORS AND FOURIER TRANSFORMS ON L^2

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1. **Introduction.** A well known theorem of Titchmarsh [**2**] states that if $f \in L^2(0, \infty)$ and if g is the Fourier cosine transform of f, then $G(x) = x^{-1}\int_0^x g(y)dy$ is the cosine transform of $F(y) = \int_y^\infty (f(x)/x)dx$ (both F and G being in L^2). The same result applies to sine transforms.

In this paper we prove the following result for a wide class of functions ψ: If g is the cosine transform of $f \in L^2$ then

$$G(x) = x^{-1}\int_0^\infty \psi(y/x)g(y)dy$$

is the cosine transform of $F(y) = \int_0^\infty x^{-1}\psi(y/x)f(x)dx$. (The same result again applies to sine transforms.) The theorem of Titchmarsh stated above is the special case of our result in which ψ is the characteristic function of $(0, 1)$.

We shall prove the above result by developing properties of a certain class of bounded operators on L^2.

Finally we shall construct a class of self-adjoint bounded operators which commute with the Fourier cosine (or sine) transform.

2. **Preliminaries.** We shall denote $L^p(0, \infty)$ by L^p, $(p = 1, 2)$ with the L^p norm $\|f\|_p$ defined as usual as $(\int_0^\infty |f(x)|^p dx)^{1/p}$. If T is a linear transformation on L^2 into itself then $\|T\|$ is defined as

$$\operatorname*{lub}_{g \in L^2} \|Tg\|_2/\|g\|_2.$$

We shall make use of the

Schwarz Inequality: if $f, g \in L^2$ then $fg \in L'$ and $\|fg\|_1 \leqq \|f\|_2\|g\|_2$, and its

Converse: if for each $h \in L^2$, $\|Gh\|_1 \leqq A\|h\|_2$ then $G \in L^2$ and $\|G\|_2 \leqq A$.

3. **A Class of bounded operators on L^2.**

Lemma. *If $\psi(y) \geqq 0$ and $\int_0^\infty \psi(y)y^{-1/2}dy = A < \infty$ then for any $g, h \in L^2$*

$$\int_0^\infty \psi(y)dy \int_0^\infty |h(x)g(xy)|\,dx \leqq A\|h\|_2\|g\|_2.$$

Proof. For $y > 0$

Received by the editors October 30, 1958.

$$\int_0^\infty |g(xy)|^2 dx = \frac{1}{y}\int_0^\infty |g(x)|^2 dx.$$

Therefore, by the Schwarz Inequality,

$$\int_0^\infty |h(x)g(xy)|\, dx \leqq \|h\|_2 \cdot \frac{1}{y^{1/2}} \|g\|_2.$$

Hence

$$\int_0^\infty \psi(y)dy \int_0^\infty |h(x)g(xy)|\, dx \leqq \|h\|_2\|g\|_2 \int_0^\infty \psi(y)y^{-1/2}dy = A\|h\|_2\|g\|_2.$$

The first part of the next theorem was proved in a much different form by Schur [1].

THEOREM 1. *Let ψ be non-negative with $\int_0^\infty \psi(y)y^{-1/2}dy = A < \infty$. Let ψ define the linear transformation T on L^2 as follows:*

$$Tg = G \text{ means } G(x) = \frac{1}{x}\int_0^\infty \psi\left(\frac{y}{x}\right) g(y)dy \qquad (g \in L^2).$$

Then T is a bounded operator on L^2 and $\|T\| \leqq A$.

Furthermore if we define T^* as

$$T^*f = F \text{ means } F(x) = \int_0^\infty \frac{1}{y}\psi\left(\frac{x}{y}\right) f(y)dy \qquad (f \in L^2),$$

then T^* is the adjoint of T and so $\|T^*\| \leqq A$.

PROOF. We shall first show that $G \in L^2$ and that $\|G\|_2 \leqq A\|g\|_2$.

For any $h \in L^2$ we have

$$\int_0^\infty |G(x)h(x)|\, dx \leqq \int_0^\infty \frac{|h(x)|}{x} dx \int_0^\infty \psi\left(\frac{y}{x}\right) |g(y)|\, dy$$

$$= \int_0^\infty |h(x)|\, dx \int_0^\infty \psi(y)\, |g(xy)|\, dy = \int_0^\infty \psi(y)dy \int_0^\infty |h(x)g(xy)|\, dx.$$

The last iterated integral converges (absolutely) by the lemma justifying the change in order of integration. Thus by the lemma

$$\|Gh\|_1 \leqq A\|g\|_2\|h\|_2.$$

The converse of the Schwarz Inequality thus implies that

$$G \in L^2 \quad \text{and} \quad \|G\|_2 \leqq A\|g\|_2.$$

Since $G = Tg$ this shows that $\|Tg\|_2 \leqq A\|g\|_2$ for all $g \in L^2$ and so T is a bounded linear transformation on L^2 into itself (bounded operator) and $\|T\| \leqq A$. The first part of the theorem is thus established.

Now choose any $f, g \in L^2$. Then with (a, b) defined as $\int_0^\infty a(x)b(x)dx$, the usual inner product in L^2, we have

$$(Tg, f) = \int_0^\infty \frac{f(x)}{x}\,dx \int_0^\infty \psi\left(\frac{y}{x}\right) g(y)dy, \tag{1}$$

and

$$(g, T^*f) = \int_0^\infty g(y)dy \int_0^\infty \frac{1}{x}\,\psi\left(\frac{y}{x}\right) f(x)dx. \tag{2}$$

The integrals in (1) and (2) converge absolutely by the lemma and hence are equal. Thus

$$(Tg, f) = (g, T^*f)$$

which, by definition of adjoint, shows that T^* is the adjoint of T. Finally, since $\|T^*\| = \|T\|$, we have $\|T^*\| \leqq A$ and the proof is complete.

In passing we remark that the integrals defining F and G in the statement of Theorem 1 exist only almost everywhere.

4. **Relation to Fourier transforms.** We shall write $Uf = g$ if g is the Fourier cosine transform of f. Thus if $Uf = g$ then

$$g(y) = \underset{R\to\infty}{\text{l.i.m.}} \left(\frac{2}{\pi}\right)^{1/2} \int_0^R f(t) \cos yt dt \qquad f \in L^2,$$

where l.i.m. stands for limit in the L^2 mean. Furthermore

$$g(y) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty f(t) \cos yt dt \qquad \text{if } f \in L' \cap L^2,$$

the above holding for almost all y.

It is well known that if $f \in L^2$ and $Uf = g$ then $g \in L^2$ and $Ug = f$. Moreover U is a self-adjoint operator ($U = U^*$).

It will be readily verified that everything we prove about the Fourier cosine transform U will also hold for the Fourier sine transform.

THEOREM 2. *If ψ is non-negative, $\psi \in L'$, and $\int_0^\infty \psi(y) y^{-1/2} dy < \infty$ then*

$$TU = UT^*$$

where T, T^* are as in Theorem 1.

PROOF. It is sufficient to prove

$$TUf = UT^*f \quad \text{for} \quad f \in L' \cap L^2$$

since $L' \cap L^2$ is dense in L^2 and T, T^*, U are continuous on L^2.

Accordingly, choose any $f \in L' \cap L^2$ and let

$$g = Uf, \qquad G = Tg, \qquad F = T^*f.$$

We need only show that $G = UF$. With $c = (2/\pi)^{1/2}$ we have

$$
\begin{aligned}
G(x) &= \frac{1}{x}\int_0^\infty \psi\left(\frac{y}{x}\right) g(y)dy = \frac{c}{x}\int_0^\infty \psi\left(\frac{y}{x}\right) dy \int_0^\infty f(t)\cos ytdt \\
&= \frac{c}{x}\int_0^\infty f(t)dt \int_0^\infty \psi\left(\frac{y}{x}\right)\cos ytdy \\
&= c\int_0^\infty f(t)dt \int_0^\infty \psi(y)\cos xytdy \qquad (3)\\
&= c\int_0^\infty \frac{f(t)}{t}dt \int_0^\infty \psi\left(\frac{y}{t}\right)\cos xydy \\
&= c\int_0^\infty \cos xydy \int_0^\infty \frac{1}{t}\psi\left(\frac{y}{t}\right) f(t)dt \\
&= c\int_0^\infty F(y)\cos xydy.
\end{aligned}
$$

The integral in (3) converges absolutely since $\psi, f \in L'$. This justifies the changes in order of integration and also shows that $F \in L'$. Thus $G = UF$ which is what we wished to show.

REMARK. If we set

$$
\begin{aligned}
\psi(y) &= 1, \qquad 0 \leqq y \leqq 1; \\
\psi(y) &= 0, \qquad y > 1,
\end{aligned}
$$

then if $G = Tg$, $F = T^*f$ we have

$$G(x) = \frac{1}{x}\int_0^x g(y)dy, \qquad F(y) = \int_y^\infty \frac{f(x)}{x}dx.$$

From Theorem 2 we see that if $g = Uf$ then $G = UF$. This is the theorem of Titchmarsh mentioned in the introduction.

5. A more general result. We may drop the hypothesis that $\psi \in L'$ in Theorem 2. To see this choose any non-negative ψ such that $\int_0^\infty \psi(y)y^{-1/2}dy = A < \infty$ (but not necessarily such that $\psi \in L'$). For $n = 1, 2, \cdots$ define

$$\psi_n(y) = \psi(y), \qquad 1/n \leqq y \leqq n;$$
$$\psi_n(y) = 0, \qquad 0 \leqq y < 1/n;\ n < y < \infty.$$

Then $\int_0^\infty \psi_n(y) y^{-1/2} dy = A_n < \infty$ and, by the Lebesgue convergence theorem,

$$\lim_{n\to\infty} A_n = A.$$

Moreover if T, T_n are defined by ψ, ψ_n as in Theorem 1 then $T - T_n$ is defined by $\psi - \psi_n$ and thus, by Theorem 1,

$$\|T - T_n\| \leqq A - A_n \to 0 \qquad \text{as } n \to \infty. \tag{4}$$

But ψ_n obeys the hypotheses of Theorem 2. Hence

$$T_n U = U T_n^*.$$

Letting $n \to \infty$ and using (4) we have

$$TU = UT^*.$$

We have thus shown that $TU = UT^*$ even for T defined by a non-negative ψ for which we assume only $\int_0^\infty \psi(y) y^{-1/2} dy < \infty$. We now state this in detail.

THEOREM 3. *Let ψ be non-negative with $\int_0^\infty \psi(y) y^{-1/2} dy < \infty$. Define the linear transformation T on L^2 as follows:*

$$Tg = G \text{ means } G(x) = \frac{1}{x} \int_0^\infty \psi\left(\frac{y}{x}\right) g(y) dy.$$

Then T is a bounded operator on L^2. Moreover if T^* is the adjoint of T and U is the Fourier cosine transform then

$$TU = UT^*.$$

REMARK. This theorem, translated back into classical terminology, is the generalization of the theorem of Titchmarsh stated in the introduction.

6. Operators that commute with the cosine transform. In order for T to be self-adjoint ($T = T^*$) we see from the definition of T, T^* in Theorem 1 that it is sufficient to have

$$\frac{1}{x} \psi\left(\frac{y}{x}\right) = \frac{1}{y} \psi\left(\frac{x}{y}\right), \qquad 0 < x, y < \infty;$$

or

$$\psi(y) = \frac{1}{y}\psi\left(\frac{1}{y}\right), \qquad 0 < y < \infty. \tag{5}$$

Suppose then that we have a non-negative function ψ defined on $(0, 1]$ such that

$$\int_0^1 \psi(y)y^{-1/2}dy < \infty \tag{6}$$

and define $\psi(y)$ for $y>1$ by

$$\psi(y) = \frac{1}{y}\,\psi\left(\frac{1}{y}\right), \qquad 1 < y < \infty.$$

Then if $y_1<1$ we have

$$\psi\left(\frac{1}{y_1}\right) = y_1\psi(y_1)$$

so that $\psi(y)=(1/y)\psi(1/y)$ for all $y>0$ (i.e. (5) holds). From (5) and (6) we have

$$\int_1^\infty \psi(y)y^{-1/2}dy = \int_1^\infty \psi\left(\frac{1}{y}\right)y^{-3/2}dy = \int_0^1 \psi(y)y^{-1/2}dy < \infty.$$

This and (6) imply

$$\int_0^\infty \psi(y)y^{-1/2}dy < \infty$$

so that the hypotheses of Theorem 3 hold. From (5) we conclude that the T defined by ψ is self-adjoint so that we have the following consequence of Theorem 3.

THEOREM 4. *Let ψ be non-negative on* $(0, 1]$ *with* $\int_0^1\psi(y)y^{-1/2}dy<\infty$. *Define* $\psi(y)=(1/y)\psi(1/y)$ *for* $y>1$. *Then if T is as in Theorem* 1

$$TU = UT.$$

In other words T commutes with the Fourier cosine transform.

REFERENCES

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