METRISABILITY OF UNIONS OF SPACES

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1. Introduction. Suppose a topological space S is expressed as the union of a family of subspaces S_α, each of which is metrisable; what further conditions will guarantee that S is itself metrisable? The following results are well known; they are due to Nagata and Smirnov [5; 7].

(A) If S is the union of a locally finite system of closed metrisable subspaces S_α, then S is metrisable.

(B) If S is a locally countably compact Hausdorff space which is the union of a sequence of separable metrisable spaces S_n ($n = 1, 2, \cdots$), then S is metrisable.

Our main object is to extend (B) to deal with nonseparable spaces. This extension is not quite straightforward, even when there are only two subspaces S_α, as is shown by the following simple example (which nullifies many conjectures in this field). Let S_1 be an uncountable discrete set, S its 1-point compactification by a 1-point set S_2; both S_1 and S_2 are metrisable, but their union S is not, though it is compact Hausdorff, and moreover S_1 is open and S_2 is closed in S. We shall therefore assume in what follows that the sets S_α are either all closed, or all open, in S. Besides extending (B) to the case of nonseparable closed subsets, we give an alternative proof of a slightly sharpened form of (A), and obtain some analogues of (A) for open sets S_α, the hypothesis of local finiteness being tightened or relaxed.

2. Closed sets.

THEOREM 1. *Let S be a collectionwise normal, locally countably compact space which is the union of a sequence of closed subspaces S_n ($n = 1, 2, \cdots$), each of which is metrisable. Then S is metrisable.*

S is a T_1 space, for each point of S is closed in some S_n, and so in S. Hence (being normal) S is regular.

Each S_n has a σ-discrete basis of relatively open sets $V_n(m, \lambda)$; here $m = 1, 2, \cdots$, λ ranges over some index set, and for fixed n and m the collection $\{V_n(m, \lambda)\}$ is discrete (in S_n). Because S_n is closed in S, the collection $\{V_n(m, \lambda)\}$ is also discrete in S; as S is collectionwise normal, a remark of Dowker [3, p. 308] shows the existence of open subsets $G_n(m, \lambda)$ of S which satisfy: $G_n(m, \lambda) \supset \overline{V}_n(m, \lambda)$ and the

collection $\{\overline{G}_n(m, \lambda)\}$ is discrete (in S), again for fixed m and n. Now there exist open sets $U_n(m, \lambda)$ in S such that $V_n(m, \lambda) = S_n \cap U_n(m, \lambda)$, and we may assume $U_n(m, \lambda) \subset G_n(m, \lambda)$; furthermore we take $U_n(m, \lambda)$, where possible, so that its closure is countably compact. Where this is not possible, we simply omit $V_n(m, \lambda)$—i.e., we discard the corresponding value of λ. Since S is locally countably compact, the sets $V_n(m, \lambda)$ which are retained still form (for each n) a basis for S_n. Thus, without loss of generality, we may assume $V_n(m, \lambda) = S_n \cap U_n(m, \lambda)$, $\{\overline{U}_n(m, \lambda)\}$ a discrete collection (for fixed m, n) of countably compact sets.

Keeping $m_0, n_0, \lambda_0, m, n$ fixed for the moment, we consider those λ's for which $\overline{V}_n(m, \lambda) \cap \overline{V}_{n_0}(m_0, \lambda_0) = \varnothing$. Using the normality of S, we pick open sets[1] $W(\lambda) \supset V_n(m, \lambda)$ so that $\overline{W}(\lambda) \cap \overline{V}_{n_0}(m_0, \lambda_0) = \varnothing$ and $W(\lambda) \subset U_n(m, \lambda)$. Write

$$F = F(m_0, n_0, \lambda_0, m, n) = \bigcup \overline{W}(\lambda),$$

the union being taken over all λ's for which $W(\lambda)$ has been defined. Then F is closed, because the collection $W(\lambda)$ is locally finite (refining $\{U_n(m, \lambda)\}$). Hence, for each positive integer N, the set $X_{n_0}(m_0, \lambda_0, N) = \bigcup \{F \mid m, n \leq N\}$ is closed.

Now consider all sets of the form $U_{n_0}(m_0, \lambda_0) - X_{n_0}(m_0, \lambda_0, N)$. They are open, and form a σ-discrete collection since for fixed m_0, n_0, N they refine the discrete collection $\{U_{n_0}(m_0, \lambda)\}$. We show they form a basis in S. Given an open neighborhood G of a point p in S, we have $p \in S_{n_0}$ for some n_0. There is an open set U such that $p \in U$ and $\overline{U} \subset G$, and there is some basic set $V_{n_0}(m_0, \lambda_0)$ containing p and included in U. It will be enough to prove that, for some N,

$$U_{n_0}(m_0, \lambda_0) - X_{n_0}(m_0, \lambda_0, N) \subset G,$$

for this set certainly contains p (since $X_{n_0}(m_0, \lambda_0, N)$ is a union of sets $\overline{W}(\lambda)$ none of which meets $V_{n_0}(m_0, \lambda_0)$). Suppose not; then, for each N, there is a point $q_N \in U_{n_0}(m_0, \lambda_0) - X_{n_0}(m_0, \lambda_0, N)$ such that $q_N \notin G$. As $\overline{U}_{n_0}(m_0, \lambda_0)$ is countably compact, the sequence q_N has a cluster point r in $\overline{U}_{n_0}(m_0, \lambda_0)$. Now $r \notin \overline{V}_{n_0}(m_0, \lambda_0)$, as otherwise $r \in \overline{U} \subset G$, and some q_N would be in G. Hence, if $r \in S_n$ say, there is some $V_n(m, \lambda) \ni r$ such that $\overline{V}_n(m, \lambda) \cap \overline{V}_{n_0}(m_0, \lambda_0) = \varnothing$. There is then a corresponding open set $W(\lambda)$, and we have $r \in W(\lambda) \subset F \subset X_{n_0}(m_0, \lambda_0, N)$ whenever $N > \max(m, n)$. The neighborhood $W(\lambda)$ of r must contain points q_N for arbitrarily large N, and then we obtain $q_N \in X_{n_0}(m_0, \lambda_0, N)$, giving a contradiction.

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[1] $W(\lambda)$ also depends on $m_0, n_0, \lambda_0, m, n$, but to simplify the notation we do not write this dependence explicitly. Similar abbreviations are used elsewhere in the proof.

Thus S has a σ-discrete basis, and is regular; by the Nagata-Smirnov theorem (see [5; 6]), S is metrisable.

COROLLARY 1. *If S is collectionwise normal and locally countably compact, and is the union of a σ-locally finite system of closed metrisable subspaces, then S is metrisable.*

For if $S = \cup S(n, \lambda)$ here, where $n = 1, 2, \cdots$, and for fixed n the collection $\{S(n, \lambda)\}$ is locally finite, we write $S_n = \cup_\lambda S(n, \lambda)$, a closed set; by (A), S_n is metrisable, and by Theorem 1 so is S.

COROLLARY 2. *If S is a paracompact Hausdorff space which is locally countably compact, and S is the union of a locally countable system of closed metrisable subspaces, then S is metrisable.*

For an application of Theorem 1 to a suitable neighborhood of a general point shows that S is locally metrisable; thus S has an open covering by metrisable subsets, and this covering has a locally finite closed refinement, to which (A) applies.

REMARK. I do not know if collectionwise normality can be replaced by normality in Theorem 1, but a space given by Dieudonné [2] shows that complete regularity would not be enough. The hypothesis of local countable compactness is not superfluous, as can be seen as follows. If S is obtained from the set R of rational points of the plane by identifying the points on the y-axis to a single point, then S is collectionwise normal because it is the image of R under a closed mapping, and it is the union of a sequence of 1-point sets; but [8, Theorem 1] shows that S is not metrisable.

3. Open sets. We first consider the case in which S is the union of a *finite* number n of open metrisable sets; it is, of course, enough to deal with the case $n = 2$, and, as the results can be stated more neatly in this case, we confine ourselves to it.

LEMMA. *A necessary and sufficient condition for S to be metrisable, where $S = S_1 \cup S_2$ and S_1, S_2 are open and metrisable, is that Fr (S_1), Fr (S_2) can be enclosed in disjoint open sets.*

Necessity is obvious, for Fr (S_1) and Fr (S_2) are disjoint closed sets. Conversely, if Fr $(S_i) \subset U_i$ $(i = 1, 2)$ where U_1, U_2 are open and disjoint, then $S_1 - (U_1 \cap S_2)$ and $S_2 - (U_2 \cap S_1)$ are closed metrisable sets covering S; the result follows from (A).

Thus, under these conditions, S is metrisable if and only if it is normal. An example due to Bing [1, Example B] shows that this requirement is not vacuous, even when S is regular.

THEOREM 2. *If a regular space S is a union of two open metrisable subspaces S_1, S_2, and if either* (i) *one at least of S_1, S_2 is separable, or* (ii) Fr (S_1) *and* Fr (S_2) *are both separable, or* (iii) *one at least of* Fr (S_1), Fr (S_2) *is compact, then S is metrisable.*[2]

The assertions under (ii) and (iii) follow from the lemma, since by standard techniques we can enclose Fr (S_1) and Fr (S_2) in disjoint open sets. (Note that Fr $(S_1) \subset S_2$, so that if it is separable it has a countable base.) To deal with (i), suppose that S_2 is separable, and let $\{U_{n\lambda}\}$ $(n=1, 2, \cdots)$ be a σ-discrete open basis for S_1, the sets $U_{n\lambda}$ for fixed n forming a discrete collection $\mathfrak{U}_n$ relative to S_1. At most countably many of the sets of $\mathfrak{U}_n$ can meet S_2; these we enumerate as V_{n1}, V_{n2}, $\cdots$, forming the subcollection $\mathfrak{V}_n$ of $\mathfrak{U}_n$. The family $\mathfrak{U}_n - \mathfrak{V}_n$ is discrete in S. Let $\{W_n\}$ be a countable open basis for S_2. Then the systems of sets $\mathfrak{U}_n - \mathfrak{V}_n$ $(n=1, 2, \cdots)$, together with the countably many 1-element families $\{V_{nm}\}$, $\{W_n\}$, form a σ-discrete basis for S, which is therefore metrisable.

The analogous results for unions of infinitely many open metrisable sets S_α seem to be less simple; we give three results of this type. Of course, in view of (A), the metrisability of $S = \cup S_\alpha$ is equivalent to the paracompactness (plus normality) of S, or to countable paracompactness and normality when the family $\{S_\alpha\}$ is countable.

THEOREM 3. *Let a regular space S be the union of a sequence of open metrisable sets S_n $(n=1, 2, \cdots)$, each of which has a compact frontier; then S is metrisable.*[3]

Let $\epsilon > 0$ be given, and let n be a positive integer, fixed for the moment. A finite number of sets $S_{m_1}, \cdots, S_{m_k}$ (with $m_i \neq n$) cover Fr (S_n). Let U_{m_i} denote the ϵ-neighborhood of Fr $(S_n) \cap S_{m_i}$ in S_{m_i}, using an arbitrarily chosen metric ρ_{m_i} for S_{m_i}; this is open in S_{m_i}, and so in S. Thus, writing $U(\epsilon) = \cup U_{m_i}$ $(i=1, \cdots, k)$, we have that $U(\epsilon)$ is open and contains Fr (S_n). It is easy to see that $\cap \{U(\epsilon) \mid \epsilon > 0\}$ $=$ Fr (S_n). Hence, using the regularity of S and the compactness of Fr (S_n), we construct recursively a sequence of open sets V_{n1}, $V_{n2}, \cdots$, such that Fr $(S_n) \subset V_{nm}$ and $\overline{V}_{n,m+1} \subset V_{nm} \cap U(1/m)$; then $\cap \{V_{nm} \mid m=1, 2, \cdots\} =$ Fr (S_n). Write $F_{nm} = S_n - V_{nm} = \overline{S}_n - V_{nm}$, a closed set interior to $F_{n,m+1}$. Fixing n and m, we cover F_{nm} by open sets contained in $F_{n,m+1}$ and of diameter less than $1/h$ (in an arbitrary

[2] "Regular" is taken to include T_1. It can, of course, be replaced by "Hausdorff" if both frontiers are compact.

[3] More generally, the sequence S_n could be replaced by any σ-locally finite system $S_{n\alpha}$ (of open metrisable sets with compact frontiers); the proof applies virtually unchanged. A similar remark applies to Theorem 4 below.

metric for S_n), h being another fixed integer; adjoining $S_n - F_{nm}$ we obtain an open covering of S_n, which has a locally finite refinement. Discarding those sets of it which fail to meet F_{nm}, we have a system $\mathfrak{U}_{nmh}$ of open sets, covering F_{nm} and contained in $F_{n,m+1}$, and locally finite in S_n and so in S. Now we let n, m, h run over all positive integers; the system $\bigcup \mathfrak{U}_{nmh}$ is easily seen to be σ-locally finite basis for S, which is therefore metrisable.

A similar, but simpler argument, proves:

THEOREM 4. *A normal space which is the union of countably many open F_σ sets, each of which is metrisable, is metrisable.*

THEOREM 5. *Let S be a regular topological space which is the union of a point-countable system of open sets S_α, each of which is locally separable and metrisable. Then S is metrisable (and locally separable).*

We form a disjoint collection of sets T_α homeomorphic to S_α, metrise each T_α to have diameter at most 1, and extend these metrics to one of $T = \bigcup T_\alpha$ by taking points in distinct T_α's to have distance 1. The natural mapping f of T onto S is continuous and open; T is metric and locally separable; and, for each $p \in S$, $f^{-1}(p)$ is countable and so separable. Hence [8, Theorem 4] applies, and S is metrisable.

The example due to Bing, mentioned earlier [1, Example B], shows that the separability requirements cannot be omitted from Theorem 5.

In conclusion, we remark that (A) may be proved by the same method as that used for Theorem 5. The space T is defined in the same way; the mapping f is now closed, and the conclusion is obtained from [4] or [8, Theorem 1]. The method gives in fact a slight weakening of the hypotheses; instead of being locally finite, it is enough that $\{S_\alpha\}$ satisfy: (1) for each choice of closed $E_\alpha \subset S_\alpha$, $\bigcup E_\alpha$ is closed, (2) $\{S_\alpha'\}$ is point-finite.[4] Little is lost if we replace (1) and (2) by the following single simpler condition (3): *Each $p \in S$ has a neighborhood meeting only finitely many sets $S_\alpha - (p)$*. (In fact, (3) implies (1) and (2), and when S is metrisable (or "first countable") (1) implies (3).) Thus condition (3) can replace local finiteness in (A) and in Corollary 1 to Theorem 1, as well as in some other applications.

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[4] E' denotes the set of limit (accumulation) points of $E \subset S$.

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CORRESPONDANCE ENTRE DEUX SURFACES PAR DES FAISCEAUX DE TANGENTES PARALLÈLES

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1. En continuant nos recherches relatives à la correspondance par parallélisme des plans tangents des deux surfaces, nous nous sommes proposé d'établir le nombre maximum possible des couples de tangentes parallèles. D'après les résultats obtenus antérieurement il sembla que ce nombre maximum doit être quatre. Mais nous avons découvert recemment que cela n'est pas ainsi et qu'il existe une catégorie spéciale de correspondances, par une double infinité de couples de tangentes parallèles.

2. Nous notons comme d'habitude les deux surfaces, rapportées au même système de paramètres curvilignes u, v, par $S(x, y, z)$, $\overline{S}(\bar{x}, \bar{y}, \bar{z})$ et nous utillisons les notations vectorielles usuelles r, r_u, r_v, $\cdots$ pour représenter les coordonnées x, y, z et leurs dérivées $\partial x/\partial u$, $\cdots$, $\partial x/\partial v$, $\cdots$.

Nous considérons le couple de deux surfaces S, $\overline{S}$, liées par les relations

(1)
$$\bar{r}_u = \lambda r_u,$$
$$\bar{r}_v = \mu r_v.$$

Il est aisé de voir que les tangentes aux lignes u, v, sur la première