ON THE ZEROS OF INFRAPOLYNOMIALS FOR PARTLY ARBITRARY POINT SETS[1]

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1. **Introduction.** Let S be a compact point set and let $p(z)=z^n+a_1z^{n-1}+\cdots+a_n$ be an infrapolynomial on S in the sense that it has no underpolynomial on S. (A polynomial $q(z)=z^n+b_1z^{n-1}+\cdots+b_n$ is said to be an underpolynomial of $p(z)$ on S if $q(z)=0$ on the subset S' of S where $p(z)=0$ and if $|q(z)|<|p(z)|$ on $S-S'$.) Fejér [1][2] has proved that all the zeros of $p(z)$ lie in the convex hull $K(S)$ of S. His theorem is analogous to the theorem of Lucas[3] that the zeros of the derivative of a polynomial $f(z)$ lie in the convex hull of the set of the zeros of $f(z)$.

Various other theorems have been proved concerning the location of the zeros of an infrapolynomial. Like Fejer's theorem, many are analogous to known theorems on the location of *all* the zeros of the derivative of a polynomial $f(z)$ when the positions of *all* the zeros of $f(z)$ are prescribed.

However, in the literature there are a number of theorems concerning the location of *some* of the zeros of the derivative of a polynomial $f(z)$ when the positions of only *some* of the zeros of $f(z)$ are prescribed. An example is the theorem of Grace and Heawood[4] that, if z_1 and z_2 are any two distinct zeros of an nth degree polynomial $f(z)$, at least one zero of the derivative of $f(z)$ lies in the circle with center at $(1/2)(z_1+z_2)$ and radius $(1/2)|z_1-z_2|\cot(\pi/n)$. Another example is the theorem due to Marden[5] that, if K zeros $(2\leqq K\leqq n)$ of an nth degree polynomial $f(z)$ lie in a circle of radius R, at least $K-1$ zeros of $f'(z)$ lie in the concentric circle of radius $R\csc[\pi/2(n-K+1)]$.

It is the object of the present note to develop for infrapolynomials results analogous to these theorems of Grace-Heawood and Marden. The results involve the location of *some* of the zeros of an infrapoly-

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[2] A number enclosed in a square bracket refers to the bibliography at the close of this note.

[3] See [**4**, pp. 14–16].

[4] See [**4**, pp. 84–85].

[5] See [**3**, p. 364] or [**4**, p. 90].

nomial for a set S comprised of points, only *some* of which have prescribed positions.

2. THEOREM 1. *Let $p(z)=z^n+a_1z^{n-1}+\cdots+a_n$ be an infrapolynomial on the set $S=S_0+S_1$ where S_0 is a compact point set (finite or infinite) and S_1 is a set of k points, $0\leqq k\leqq n$. Let T_0 be the set comprised of all points from which S_0 subtends an angle of at least $\pi/(k+1)$. If $p(z)\neq 0$ on S, then $p(z)$ has at most k zeros outside T_0 irrespective of the location of S_1.*

PROOF. We shall make use of the following result due to Fekete [2].

FEKETE'S THEOREM. *Let $p(z)=z^n+a_1z^{n-1}+\cdots+a_n$ be an infrapolynomial on S. If $p(z)\neq 0$ on S, then there exist $(m+1)$ positive constants λ_j with $\lambda_0+\lambda_1+\cdots+\lambda_m=1$ and $(m+1)$ points z_j in S such that $p(z)$ is a factor of the polynomial*

$$\Omega(z) = \omega(z)\sum_{j=0}^{m}[\lambda_j/(z-z_j)], \qquad n \leqq m \leqq 2n,$$

where

$$\omega(z) = (z-z_0)(z-z_1)\cdots(z-z_m).$$

It follows that the zeros of $p(z)$ satisfy an equation of the form

$$\sum_{j=0}^{m}\lambda_j/(z-z_j) = 0. \tag{2.1}$$

Let us suppose that $p(z)$ has $k+1$ zeros, $Z_0, Z_1, \cdots, Z_k$ outside T_0. Let $(2h)$ be the shortest distance of any of the points Z_j to the boundary of T_0. The zeros of (2.1) are continuous functions of the λ_j. If we choose ϵ as a sufficiently small number, $0<\epsilon<h$, we can find a $\delta>0$ such that for rational numbers ρ_j with $|\rho_j-\lambda_j|<\delta$, $j=0, 1, \cdots, m$, the equation

$$\sum_{j=0}^{m}[\rho_j/(z-z_j)] = 0 \tag{2.2}$$

has roots $\zeta_0, \zeta_1, \cdots, \zeta_k$ with $|\zeta_j-Z_j|<\epsilon$ for $j=0, 1, \cdots, m$. Thus the points $\zeta_0, \zeta_1, \cdots, \zeta_k$ also lie outside T_0.

Let us next choose a positive, sufficiently large integer N so that each of the quantities $\nu_j=\rho_jN$ is an integer. The equation

$$\sum_{j=0}^{m}[\nu_j/(z-z_j)] = 0 \tag{2.3}$$

which has also the roots $\zeta_0, \zeta_1, \cdots, \zeta_k$, is satisfied by those zeros of the derivative of the polynomial $f(z)=\prod_{j=0}^{m}(z-z_j)^{\nu_j}$ which are distinct from the z_j.

Since all that is known about the z_j is that each lies in S, some of them may be points of S_1. But the set S_1 contains only k points so that at least $m+1-k$ of the z_j do not lie in S_1. Let us, if necessary, relabel the z_j so that the points $z_0, z_1, \cdots, z_{m-k}$ all lie in S_0. Now any set of $m+2$ distinct points comprised of $m-k+1$ zeros $z_0, z_1, \cdots, z_{m-k}$ of $f(z)$ and $k+1$ zeros $\zeta_0, \zeta_1, \cdots, \zeta_k$ of $f'(z)$, are known[6] to satisfy an identity of the form

$$(2.4)\qquad \sum M_{j_0 j_1 \cdots j_k}[(\zeta_0 - z_{j_0})(\zeta_1 - z_{j_1}) \cdots (\zeta_k - z_{j_k})]^{-1} = 0$$

where the $j_0, j_1, \cdots, j_k$ run independently through the values $0, 1, \cdots, m-k$ and the numbers $M_{j_0 j_1 \cdots j_k}$ are all positive.

Since $\zeta_0, \zeta_1, \zeta_k$ all lie outside T_0, points $\tau_0, \tau_1, \cdots, \tau_k$ may be found in S_0 so that for each i

$$(2.5)\qquad 0 < \arg\left[(\zeta_i - \tau_i)/(\zeta_i - z_{j_i})\right] < \pi/(k+1).$$

If the left-side of (2.4) is multiplied by $(\zeta_0-\tau_0)(\zeta_1-\tau_1)\cdots(\zeta_k-\tau_k)$, each term in the resulting sum has, due to (2.5), the property

$$0 < \arg \prod_{i=0}^{k}\left[(\zeta_i - \tau_i)/(\zeta_i - z_{j_i})\right] < \pi.$$

Hence the sum in (2.4) cannot vanish and so at most k zeros of any infrapolynomial for S can lie outside of T_0.

3. **Corollaries.** Fejér's Theorem, stated in §1, is the special case of Theorem 1 corresponding to $k=0$. Other immediate corollaries are the following:

COROLLARY 1. *Let S_0 be a compact point set all of the points of which lie in a circle C of radius R. Let S be the point set comprised of S_0 and k additional points, the latter having arbitrary positions in the complex plane. Then any infrapolynomial $p(z)=z^n+a_1z^{n-1}+\cdots+a_n\neq 0$ on S has at least $n-k$ zeros within the concentric circle of radius $R \csc[\pi/2(k+1)]$.*

COROLLARY 2. *Let S_0 be a compact set situated on the line segment joining the points $z=a$ and $z=b$ of the real axis. Let S be comprised of S_0 and k additional points situated at random in the plane. If $p(z)=z^n+\cdots$ is any nonvanishing infrapolynomial on S, then at least $n-k$ zeros of $p(z)$ lie in region bounded by the two circles with radius $L\csc \pi/(k+1)$ and centers $z=c\pm iL\cot[\pi/(k+1)]$ where $c=(1/2)(a+b)$ and $L=(1/2)|a-b|$.*

[6] See [3, pp. 356–364].

4. **Generalization.** Theorem 1 may be extended to the case that $p(z)$ has some zeros on S, as follows.

THEOREM 2. *In the notation of Theorem* 1, *if* $p(z)$ *has* ν *zeros on* S_1 *where* $0 \leqq \nu \leqq \min(k, n-k)$, *then* $p(z)$ *has at most* $\nu+k$ *zeros* (*counted with their multiplicities*) *outside* T_0 *irrespective of the location of* S_1.

PROOF. Let us assume that $p(z)$ has zeros both on S_0 and S_1 and denote these by $\xi_1, \xi_2, \cdots, \xi_\mu$ and $\zeta_1, \zeta_2, \cdots, \zeta_\nu$ respectively where $0 \leqq \mu \leqq n$, $0 \leqq \nu \leqq n$, $\mu+\nu \leqq \min(k, n-k)$. Let us write

$$p(z) = (z - \xi_1) \cdots (z - \xi_\mu)(z - \zeta_1) \cdots (z - \zeta_\nu) p_1(z)$$

where $p_1(z) \neq 0$ on S. Then, relative to polynomials of degree $n-\mu-\nu$, $p_1(z)$ is an infrapolynomial on S, since otherwise $p_1(z)$ would have an underpolynomial $q_1(z)$ and thus $p(z)$ would have the underpolynomial $q(z) = [p(z)/p_1(z)]q_1(z)$.

On applying Theorem 1, we learn that $p_1(z)$ has at least $n-\mu-\nu-k$ zeros on T_0. But, as $S_0 \subset T_0$, $p(z)$ has as zeros in T_0 the points ξ_j as well as the zeros of $p_1(z)$. Thus $p(z)$ has at least $n-\nu-k$ zeros in T_0, as was to be proved.

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