

ON BOUNDED FUNCTIONS WITH BOUNDED n TH DIFFERENCES

HASSLER WHITNEY

We consider real valued functions f defined in a closed interval I (bounded or unbounded), with n th differences

$$\Delta_h^n f(x) = \sum_i (-1)^{n-i} \binom{n}{i} f(x + ih)$$

bounded for some fixed n .

THEOREM. *For each integer $n \geq 1$ there is a number L_n with the following property. Let f be defined in I , and suppose there is an interval $I' \subset I$ in which f is bounded. Then there is a polynomial P of degree $\leq n-1$ such that*

$$(1) \quad |f(x) - P(x)| \leq L_n \sup |\Delta_h^n f(y)|, \quad x \in I.$$

Assuming f continuous, this was proved by the author [4], and with a different definition of differences, by J. C. Burkill [1]. The present theorem, for I unbounded, was proved recently by F. John [2], who kindly showed me his manuscript. We show here that the theorem follows easily from the proof of the theorem of [4], together with a method of proof [4, §12] due to A. Beurling.

It is sufficient to consider the intervals

$$I_0 = (0, 1), \quad I^* = (0, \infty), \quad I^{**} = (-\infty, \infty).$$

The smallest constants L_n^* , L_n^{**} in the latter two cases for which (1) holds will be shown to satisfy

$$(2) \quad L_n^* \leq 1, \quad L_n^{**} \leq 1 / \sup_k \binom{n}{k}.$$

For further information about the constants, compare [4]. The inequality for L_n^{**} is in fact an equality if n is even.

It would be of interest to prove the theorem assuming merely that f is measurable (compare [3, Theorem 1]).

We consider first the interval I_0 ; we may suppose that $|\Delta_h^n f(x)| \leq 1$. Let T be the set of all numbers of the form $i/2^k(n-1)$, i and k being integers. In [4, §4], we may take ν to be a power of 2. Define the quantities in §§4, 5 and 7 of [4], in particular, K_n' (not the K_n' of

Received by the editors September 15, 1958.

§1). Choose the polynomial P_f of degree $\leq n-1$, equal to f at the points $i/(n-1)$ ($i=0, 1, \dots, n-1$), and set $g=f-P_f$. We show first that

$$(3) \quad |g(x)| \leq K'_n, \quad x \in T \cap I_0.$$

Noting that the sets S_k and T_k exactly fill out the set $T \cap I_0$, (3) is proved in [4, §7].

Let U be the open set of points x , $0 < x < 1$, such that f (and hence g) is bounded in a neighborhood of x . We show next that

$$(4) \quad |g(x)| \leq K'_n + 2, \quad x \in U.$$

Take $x \in U$; say $|g(y)| \leq K$ in the interval J about x . Take any integer N . We may choose a point $y \in J \cap T$ so close to x that if $y = x + n!h$, then all points $x + ijh$ and $y + ijh$ ($0 \leq i \leq N$, $0 \leq j \leq n$) are in J . Since $|\Delta_u^n g(v)| \leq 1$, we may apply the proof in §12 of [4], giving

$$|g(y) - g(x)| < 2 + 2^{n+1}n!K/N;$$

since $|g(y)| \leq K'_n$ and N is arbitrary, (4) follows.

To complete the proof of the theorem for I_0 (and show that $L_n \leq K'_n + 2$), we need merely show that U contains all points interior to I_0 . Suppose not. Then U contains an open interval (x_0, x_1) , x_0 (or x_1) being interior to I_0 , $x_0 \notin U$. We may choose x (near x_0) and h so that

$$|g(x)| \geq 2^n(K'_n + 2),$$

and the points $x + ih$ ($i=1, 2, \dots, n$) are in U . Then by (4),

$$|\Delta_h^n g(x)| \geq 2^n(K'_n + 2) - \sum_{i=1}^n \binom{n}{i} (K'_n + 2) > 1,$$

a contradiction.

The proof in [4, §8], now applies, giving the theorem for unbounded intervals. Applying [4, §12] now gives (2) also.

REFERENCES

1. J. C. Burkill, *Polynomial approximations to functions with bounded differences*, J. London Math. Soc. vol. 33 (1958) pp. 157-161.
2. F. John, 1958 (unpublished).
3. H. Whitney, *Derivatives, difference quotients and Taylor's formula*, Bull. Amer. Math. Soc. vol. 40 (1934) pp. 89-94.
4. ———, *On functions with bounded n th differences*, J. Math. Pures Appl. (9) vol. 36 (1957) pp. 67-95.

INSTITUTE FOR ADVANCED STUDY