ON BOUNDED FUNCTIONS WITH BOUNDED nTH DIFFERENCES

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We consider real valued functions f defined in a closed interval I (bounded or unbounded), with nth differences

$$\Delta_h^n f(x) = \sum_i (-1)^{n-i} \binom{n}{i} f(x+ih)$$

bounded for some fixed n.

THEOREM. *For each integer $n \geqq 1$ there is a number L_n with the following property. Let f be defined in I, and suppose there is an interval $I' \subset I$ in which f is bounded. Then there is a polynomial P of degree $\leqq n-1$ such that*

$$|f(x) - P(x)| \leqq L_n \sup |\Delta_h^n f(y)|, \qquad x \in I. \tag{1}$$

Assuming f continuous, this was proved by the author [**4**], and with a different definition of differences, by J. C. Burkill [**1**]. The present theorem, for I unbounded, was proved recently by F. John [**2**], who kindly showed me his manuscript. We show here that the theorem follows easily from the proof of the theorem of [**4**], together with a method of proof [**4**, §12] due to A. Beurling.

It is sufficient to consider the intervals

$$I_0 = (0, 1), \qquad I^* = (0, \infty), \qquad I^{**} = (-\infty, \infty).$$

The smallest constants L_n^*, L_n^{**} in the latter two cases for which (1) holds will be shown to satisfy

$$L_n^* \leqq 1, \qquad L_n^{**} \leqq 1 \Big/ \sup_k \binom{n}{k}. \tag{2}$$

For further information about the constants, compare [**4**]. The inequality for L_n^{**} is in fact an equality if n is even.

It would be of interest to prove the theorem assuming merely that f is measurable (compare [**3**, Theorem 1]).

We consider first the interval I_0; we may suppose that $|\Delta_h^n f(x)| \leqq 1$. Let T be the set of all numbers of the form $i/2^k(n-1)$, i and k being integers. In [**4**, §4], we may take ν to be a power of 2. Define the quantities in §§4, 5 and 7 of [**4**], in particular, K_n' (not the K_n' of

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§1). Choose the polynomial P_f' of degree $\leqq n-1$, equal to f at the points $i/(n-1)$ $(i=0, 1, \cdots, n-1)$, and set $g=f-P_f'$. We show first that

$$|g(x)| \leqq K_n', \qquad x \in T \cap I_0. \tag{3}$$

Noting that the sets S_k and T_k exactly fill out the set $T\cap I_0$, (3) is proved in [**4**, §7].

Let U be the open set of points x, $0<x<1$, such that f (and hence g) is bounded in a neighborhood of x. We show next that

$$|g(x)| \leqq K_n' + 2, \qquad x \in U. \tag{4}$$

Take $x\in U$; say $|g(y)| \leqq K$ in the interval J about x. Take any integer N. We may choose a point $y\in J\cap T$ so close to x that if $y=x+n!h$, then all points $x+ijh$ and $y+ijh$ $(0\leqq i\leqq N,\ 0\leqq j\leqq n)$ are in J. Since $|\Delta_u^n g(v)| \leqq 1$, we may apply the proof in §12 of [**4**], giving

$$|g(y) - g(x)| < 2 + 2^{n+1}n!K/N;$$

since $|g(y)| \leqq K_n'$ and N is arbitrary, (4) follows.

To complete the proof of the theorem for I_0 (and show that $L_n\leqq K_n'+2$), we need merely show that U contains all points interior to I_0. Suppose not. Then U contains an open interval (x_0, x_1), x_0 (or x_1) being interior to I_0, $x_0\notin U$. We may choose x (near x_0) and h so that

$$|g(x)| \geqq 2^n(K_n' + 2),$$

and the points $x+ih$ $(i=1, 2, \cdots, n)$ are in U. Then by (4),

$$|\Delta_h^n g(x)| \geqq 2^n(K_n' + 2) - \sum_{i=1}^{n}\binom{n}{i}(K_n' + 2) > 1,$$

a contradiction.

The proof in [**4**, §8], now applies, giving the theorem for unbounded intervals. Applying [**4**, §12] now gives (2) also.

References

1. J. C. Burkill, *Polynomial approximations to functions with bounded differences*, J. London Math. Soc. vol. 33 (1958) pp. 157–161.

2. F. John, 1958 (unpublished).

3. H. Whitney, *Derivatives, difference quotients and Taylor's formula*, Bull. Amer. Math. Soc. vol. 40 (1934) pp. 89–94.

4. ———, *On functions with bounded nth differences*, J. Math. Pures Appl. (9) vol. 36 (1957) pp. 67–95.

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