A NOTE ON PERIODIC SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS WITHOUT DAMPING

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We consider the differential equation

$$\ddot{x} + g(x) = p(t), \tag{1}$$

where, in addition to satisfying conditions which insure existence and uniqueness of solutions, $g(x)$ and $p(t)$ satisfy the following conditions:

(i) there exist positive constants τ and k such that for all t, $p(t) = p(t+\tau)$ and $|p(t)| < k$;

(ii) $g(x)$ is continuous, increasing, $g(0)=0$, and for $|x|$ sufficiently large, $|g(x)| > k$.

We show that for τ sufficiently small, Equation (1) has a solution $x(t)$ of period τ. Our method of proof consists of showing that a certain mapping of the phase plane of Equation (1) has a fixed point, not by use of the Brouwer theorem, but by a somewhat more general result concerning the index of the bounding curve of a simply-connected region relative to a vector field induced by the mapping; cf. [**1**, p. 337].

We consider Equation (1) in terms of the system:

$$\dot{x} = y, \qquad \dot{y} = p(t) - g(x), \tag{2}$$

the solutions $(x(t), y(t))$ of which define the so-called phase trajectories of the system. Let $x=a$ be the root of $g(x) = -k$, and $x=b$ be the root of $g(x)=k$. That these roots exist and are unique follows immediately from (ii). We define

$$G(x) = \int_0^x g(v)dv$$

and using (ii), we observe that the graph of $y=G(x)$ has the following properties:

(iii) it is tangent to the x-axis at the origin and is everywhere concave up;

(iv) its slope $G'(x)=g(x)$ is such that $-k<G'(x)<k$ for $a<x<b$, $G'(x)>k$ for $x>b$, and $G'(x)<-k$ for $x<a$.

Hence, there clearly exists a positive constant C_1 such that the line $y=kx+C_1$ intersects the graph of $y=G(x)$ at exactly two points whose abscissas r_2 and r_3 are such that $r_2<a<b<r_3$. Similarly there exists a positive constant C_2 such that the line $y=-kx+C_2$ intersects the graph of $y=G(x)$ at exactly two points whose abscissas r_1 and r_4 are such that $r_1<r_2<r_3<r_4$.

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We denote by Γ_1 the open arc defined by $y^2/2+G(x)=kx+C_1$, $y>0$, by Γ_2 the open arc defined by $y^2/2+G(x)=-kx+C_2$, $y<0$, and observe that the end points of these arcs are at $(r_2, 0)$, $(r_3, 0)$ and $(r_1, 0)$, $(r_4, 0)$ respectively.

LEMMA 1. *The slope of any phase trajectory of* (2) *at a point of either* Γ_1 *or* Γ_2 *is less than the slope of the curve* Γ_1 *or* Γ_2 *at that point.*

PROOF. We prove the lemma only for a point (x_0, y_0) on Γ_1; the proof for a point on Γ_2 is entirely similar, and is omitted. Clearly the slope of Γ_1 at (x_0, y_0) is given by $(k-g(x_0))/y_0$, while the slope of the trajectory of (2) at (x_0, y_0) for $t=t_0$ is given by $(p(t_0)-g(x_0))/y_0$, which is clearly less than the slope of Γ_1 there. This proves the lemma.

We now denote by P_1, P_2, and P_3 the points $(r_1, 0)$ and $(r_3, 0)$ respectively; by Q the intersection of the line $x=r_3$ with Γ_2, and by R the closed region bounded by the curves Γ_1, Γ_2 from P_1 to Q, and the line segments P_1P_2, P_3Q. We now prove another simple

LEMMA 2. *Let* (x_0, y_0) *be a point on the boundary of* R, *and let* $(x(t), y(t))$ *be the solution of* (2) *for which* $x(0)=x_0$, $y(0)=y_0$. *Then the trajectory of this solution can only leave* R *for some* $t_0\geqq 0$, *if* $r_1\leqq x(t_0) \leqq r_2$, *and* $y(t_0)=0$.

PROOF. Due to Lemma 1, we need only prove that if (x_0, y_0) is on the segment P_3Q, then the trajectory at (x_0, y_0) for some arbitrary $t=t_0>0$ passes first into R as t increases from t_0. This is obvious if (x_0, y_0) is Q since Q is on Γ_2; it is also obvious for (x_0, y_0) between P_3 and Q since then $y_0<0$, and hence $\dot{x}(t_0)=y(t_0)<0$. Finally if (x_0, y_0) is P_3, then since $y(t_0)=-g(x(t_0))+p(t_0)<0$, $y(t)<0$ for $t>t_0$, $t-t_0$ sufficiently small; hence $x(t)<x(t_0)$ also for $t>t_0, t-t_0$ sufficiently small. This essentially proves the lemma.

We now define $M=\left|2(G(a)+ka-C_2)\right|^{1/2}$, $m=(2(G(a)-ka-C_1))^{1/2}$, $d=a-r_2$, and $k=\max_{r_1\leqq x\leqq 0}(-g(x)+k)$ and prove the following:

THEOREM. *In terms of the definitions above, if* $\tau<\min(d/M, m/K)$, *then system* (2), *hence Equation* (1), *has a solution of period* τ.

PROOF. We define the mapping T of the (x, y) plane into itself as follows: $T(x_0, y_0)=(x_1, y_1)$ where $x_1=x(\tau)$, $y_1=y(\tau)$, and $(x(t), y(t))$ is the solution of (2) such that $x(0)=0$, $y(0)=0$. We show that this mapping has a fixed point; this will prove the theorem; cf. [**1**, p. 270].

We consider the vector field defined by this mapping, and will show that the index of the boundary of R relative of this field (cf. [**1**, p. 337]) is $+1$; this will prove that R contains a critical point of this field; i.e., that the mapping T has a fixed point in R.

Let (x_0, y_0) be a point of Γ_1; we show first that $T(x_0, y_0)$ is in R. For if not, there exists by Lemma 2 a smallest t_1, $0 < t_1 < \tau$, such that $y(t_1) = 0$, $r_1 \leqq x(t_1) \leqq r_2$; here as throughout this proof $(x(t), y(t))$ is the solution of (2) used in the definition of T. Suppose first $x_0 \geqq a$; then clearly for $0 < t < t_1$, $|y(t)| < M$. Since $\dot{x}(t) = y(t)$,

$$| x(t_1) - x_0 | = \left| \int_0^{t_1} y(t) dt \right| < M\tau.$$

However, $|x(t_1) - x_0| \geqq a - r_2 = d$; i.e., $d < \tau M$, a contradiction. If $r_2 < x_0 < a$, then since $\dot{y} = -g(x) + p(t) > 0$ for $x = x(t) < a$, there exists a $t_0 > 0$ such that $(x(t), y(t))$ is in R for $0 \leqq t \leqq t_0$, and $x(t_0) = a$. From here, the argument toward a contradiction proceeds as above, and we omit it. In fact, if (x_0, y_0) is on either the segment P_3Q or the curve Γ_2 from $(a, -M)$ to Q, the same argument, in essence, applies to show that $T(x_0, y_0)$ is in R.

If (x_0, y_0) is on Γ_2 while $x < a$, or on the segment P_1P_2, then if $T(x_0, y_0) = (x_1, y_1)$ is not in R, we show finally that $r_1 \leqq x_1 \leqq a$, $0 \leqq y_1 \leqq m$. In any case, there exists t_0, $0 \leqq t_0 < \tau$, such that $r_1 \leqq x(t_0) \leqq r_2$, $y(t_0) = 0$. If $x(t) < a$ for $t_0 < t < \tau$, then clearly $x(t) > r_1$ for these values of t. Assume that for t_1, $t_0 < t_1 < \tau$, we have $y(t_1) > m$. Then since $\dot{y} = -g(x) + p(t)$, we have

$$y(t_1) - y(t_0) = \int_{t_0}^{t_1} (-g(x(t)) + p(t)) dt < K\tau.$$

But $y(t_1) - y(t_0) = y(t_1) > m$; hence $m < K\tau$, a contradiction, and we conclude that $x(t_2) = a$ for some t_2, $t_0 < t_2 < \tau$, while $0 < y(t_2) < m$. Since $M > m$, and $x(t_2) - x(t_0) > d$, we easily arrive at a contradiction as before by integrating $\dot{x}(t) = y(t)$ from t_0 to t_2; we omit the details.

If the vector field $(x, y) \to T(x, y)$ has a critical point on the boundary of R, then by definition, the mapping has a fixed point there and there is nothing more to prove. We therefore assume that as (x, y) moves in a counter-clockwise circuit around the boundary of R, the vector at each point of the boundary is nonzero; since it is also a continuous function of (x, y) and in view of the location of the terminal point $T(x, y)$ for (x, y) on the boundary of R as established in the proof of this theorem, it follows that the index of this circuit relative to this field is $+1$. This completes the proof of the theorem.

Reference

1. S. Lefschetz, *Differential equations: Geometric theory*, Interscience Publishers, Inc., New York, 1957.

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