

ON A QUESTION OF PAUL LÉVY

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1. **Introduction.** A matrix $\{p_{ij}(t)\}$, $i, j = 1, 2, \dots$, $0 < t < \infty$, of functions is a (stationary) transition matrix (of a Markov chain) if and only if:

$$(1.1) \quad p_{ij}(t) \geq 0, \quad i, j = 1, 2, \dots,$$

$$(1.2) \quad \sum_r p_{ir}(t) = 1, \quad i = 1, 2, \dots,$$

and

$$(1.3) \quad \sum_r p_{ir}(t)p_{rj}(s) = p_{ij}(t+s), \quad i, j = 1, 2, \dots, \\ 0 < s < \infty, 0 < t < \infty.$$

The terms of such a transition matrix are called transition functions (of the underlying Markov chain).

Paul Lévy [2, §III.4] asks whether each such transition function, which is not measurable, must be a transition function of a Markov chain obtained by combining a Markov chain having measurable transition functions with one having a transition matrix of non-measurable functions which take only the values zero or one.

Lévy [2, Théorème II. 8.2] has shown that if $\{p_{ij}(t)\}$, $i, j = 1, 2, \dots$, is a transition matrix of measurable transition functions, then $\lim_{t \rightarrow \infty} p_{ij}(t)$ exists, and

$$(1.4) \quad \lim_{t \rightarrow \infty} p_{ij}(t) = \lim_{t \rightarrow \infty} p_{jj}(t) \text{ or } p_{ji}(t) \equiv 0, \quad i, j = 1, 2, \dots$$

Lévy has also essentially shown [2, Extension du Théorème II. 8.1] that if the transition functions are measurable, then they satisfy the hypothesis of our Theorem 2.²

We shall show (Theorem 2) that if each of the transition functions is strictly positive or identically zero, then (1.4) holds without measurability assumptions, and without assuming an affirmative answer to Lévy's question. Theorem 1 gives a weaker result than (1.4), with no conditions on the transition functions. These theorems together

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² *Added in proof.* Levy's proof is heuristic and it is necessary to assume that all states are stable to complete it. D. G. Austin has recently given a proof of this result without supposing that there are stable states (unpublished).

with the corollaries describe the limiting behavior of the transition functions which would be expected if the question raised by Lévy has an affirmative answer, and do so independently of whether or not it has an affirmative answer.

2. Results and proofs.

LEMMA. Let $P(t)$ be a transition matrix and let $Q = (q_{ij})$ be a matrix such that

- (i) $Q \leq P(t) \cdot Q$, and
- (ii) for each (i, j) there exists a sequence $\{s_k\}$ such that

$$q_{ij} = \lim_{k \rightarrow \infty} p_{ij}(s_k).$$

Then

$$q_{ij} = q_{jj} \text{ or } p_{ji}(t) \equiv 0, \quad i, j = 1, 2, \dots$$

PROOF. Let j be any integer, and let j_n be the smallest integer such that

$$(2.1) \quad q_{j_n j} = \max(q_{ij}, \dots, q_{nj}),$$

and let

$$(2.2) \quad q_j = \lim_{n \rightarrow \infty} q_{j_n j}.$$

We have, from (ii), that

$$(2.3) \quad q_{j_n j} \leq \sum_r p_{j_n r}(t) q_{rj}.$$

In (2.3) we may replace each q_{rj} , $r \neq j$ by q_j to obtain:

$$(2.4) \quad \begin{aligned} q_{j_n j} &\leq \sum_{r \neq j} p_{j_n r}(t) q_j + p_{j_n j}(t) q_{jj} \\ &= \sum_r p_{j_n r}(t) q_j + p_{j_n j}(t) q_{jj} - p_{j_n j}(t) q_j. \end{aligned}$$

Rewriting (2.4), we get

$$(2.5) \quad q_{j_n j} \leq q_j + p_{j_n j}(t) [q_{jj} - q_j].$$

(2.5) is true for all t , in particular for each term of the sequence $\{s_k\}$ corresponding to (j_n, j) [see (ii)]. Hence

$$(2.6) \quad q_{j_n j} \leq q_j + q_{j_n j} [q_{jj} - q_j].$$

Now letting $n \rightarrow \infty$, we obtain

$$(2.7) \quad q_j \leq q_j + q_j [q_{jj} - q_j].$$

Using (2.7) and the fact that $q_j \geq q_{jj} \geq 0$, we see that $q_{jj} = q_j$.

Similarly, we have, for each i ,

$$(2.8) \quad \begin{aligned} q_{jj} &\leq \sum_r p_{jr}(t)q_{rj} \\ &\leq \sum_r p_{jr}(t)q_{jj} + p_{ji}(t)q_{ij} - p_{ji}(t)q_{jj}, \end{aligned}$$

or, rearranging terms,

$$(2.9) \quad q_{jj} \leq q_{jj} + p_{ji}(t)[q_{ij} - q_{jj}].$$

From (2.9) we have that either $p_{ji}(t) \equiv 0$ or $q_{ij} = q_{jj}$.

THEOREM 1. *Assuming only that $P(t)$ is a transition matrix, we have*

$$(2.10) \quad \limsup_{t \rightarrow \infty} p_{ij}(t) \leq \limsup_{t \rightarrow \infty} p_{jj}(t),$$

$$(2.11) \quad \limsup_{t \rightarrow \infty} p_{ij}(t) = \limsup_{t \rightarrow \infty} p_{jj}(t) \text{ or } p_{ji}(t) \equiv 0,$$

$$(2.12) \quad \liminf_{t \rightarrow \infty} p_{ij}(t) \leq \liminf_{t \rightarrow \infty} p_{jj}(t),$$

$$(2.13) \quad \liminf_{t \rightarrow \infty} p_{ij}(t) = \liminf_{t \rightarrow \infty} p_{jj}(t) \text{ or } p_{ji}(t) \equiv 0.$$

PROOF. The matrix $\{\limsup_{t \rightarrow \infty} p_{ij}(t)\}$ is a matrix Q satisfying conditions (i) and (ii) of the lemma. Thus we have (2.10) and (2.11). The matrix $\{\liminf_{t \rightarrow \infty} p_{ij}(t)\}$ is also such a matrix Q . Indeed it clearly satisfies (ii). That it satisfies (i) was shown by Doob [1, Theorem 6]. An application of the lemma now gives us (2.12) and (2.13).

COROLLARY 1. *For each positive integer j , we have that $\lim_{t \rightarrow \infty} p_{ij}(t)$ exists for $i = 1, 2, \dots$, if and only if $\lim_{t \rightarrow \infty} p_{jj}(t)$ exists.*

PROOF. Doob [1, Theorem 6] has shown that $\lim_{t \rightarrow \infty} p_{ij}(t)$ exists except possibly when $\liminf_{t \rightarrow \infty} p_{jj}(t) = 0$. In this case, however, if $\lim_{t \rightarrow \infty} p_{jj}(t)$ exists, then it equals zero, so that by (2.10) $\lim_{t \rightarrow \infty} p_{ij}(t)$ exists (and equals zero).

COROLLARY 2. *Let j be any integer. Suppose that the $p_{ij}(t)$ are such that for each $i = 1, 2, \dots$,*

$$(2.14) \quad p_{ij}(t) > 0 \text{ or } p_{ij}(t) \equiv 0.$$

Let $\{s_k\} \uparrow \infty$ be a sequence such that

$$\lim_{k \rightarrow \infty} p_{jj}(s_k) = \limsup_{t \rightarrow \infty} p_{jj}(t).$$

Then, $\lim_{k \rightarrow \infty} p_{jj}(s_k + t)$ and $\lim_{k \rightarrow \infty} p_{jj}(s_k - t)$ exist and are both equal to $\lim_{k \rightarrow \infty} p_{jj}(s_k)$.

PROOF. From Theorem 1 we have that $\limsup_{t \rightarrow \infty} p_{rj}(t) = \limsup_{t \rightarrow \infty} p_{jj}(t)$ or $p_{jr}(t) \equiv 0$. We also have that

$$p_{jj}(s_k) = \sum_r p_{jr}(t) p_{rj}(s_k - t).$$

From these facts it follows that

$$\lim_{k \rightarrow \infty} p_{rj}(s_k - t) \text{ exists and equals } \lim_{k \rightarrow \infty} p_{jj}(s_k) \text{ or } p_{jr}(t) = 0, \\ r = 1, 2, \dots$$

In view of the hypothesis (2.14), this may be written

$$(2.15) \quad \lim_{k \rightarrow \infty} p_{rj}(s_k - t) = \lim_{k \rightarrow \infty} p_{jj}(s_k) \text{ or } p_{jr}(t) \equiv 0, \quad r = 1, 2, \dots$$

For $r = j$ (2.15) implies that

$$\lim_{k \rightarrow \infty} p_{jj}(s_k - t) = \lim_{k \rightarrow \infty} p_{jj}(s_k).$$

We have proved the first part of the corollary. To prove the second part, note that $p_{jj}(s_k + t) = \sum_r p_{jr}(2t) p_{rj}(s_k - t)$. The desired result follows at once from this and (2.15).

THEOREM 2. If $P(t)$ is a transition matrix such that for each $i = 1, 2, \dots$, and j fixed,

$$p_{ij}(t) > 0 \quad \text{or} \quad p_{ij}(t) \equiv 0,$$

then $\lim_{t \rightarrow \infty} p_{ij}(t)$ exists, $i = 1, 2, \dots$

PROOF. We shall modify an idea of Doob [1, Theorem 6].

Let \mathfrak{L} be the set of limiting matrices (as $t \rightarrow \infty$) of $P(t)$. Fix j as in the statement of the theorem, and let \mathfrak{M} be the set of all those $A \in \mathfrak{L}$ for which

$$a_{jj} = \limsup_{t \rightarrow \infty} p_{jj}(t).$$

Let $c = \inf_{A \in \mathfrak{M}} \sum_{n,m} 2^{-n} a_{nm}$. Let \mathfrak{N} be the set of all $A \in \mathfrak{M}$ such that $\sum_{n,m} 2^{-n} a_{nm} = c$. Since \mathfrak{L} contains its limit matrices, so does \mathfrak{M} and hence \mathfrak{N} is not void.

We note that if A is any matrix in \mathfrak{L} there exists a matrix A' in \mathfrak{N} such that $A \geq A'$. In fact:

(i) If $A \in \mathfrak{M}$ then $P(t) \cdot A \in \mathfrak{M}$. This follows immediately from Corollary 2.

(ii) If $A \in \mathfrak{N}$ then $P(t) \cdot A = A \cdot P(t) \in \mathfrak{N}$. This may be shown as follows: Choose a sequence $\{s_k\} \uparrow \infty$ such that $a_{nm} = \lim_{k \rightarrow \infty} p_{nm}(s_k)$. By Corollary 2, we have that $a_{jj} = \lim_{k \rightarrow \infty} p_{jj}(s_k) = \lim_{k \rightarrow \infty} p_{jj}(s_k - t)$, since $A \in \mathfrak{M}$. Now choose a subsequence $\{s'_k\}$ of $\{s_k\}$ such that $\lim_{k \rightarrow \infty} p_{nm}(s'_k - t)$ exists for $n, m = 1, 2, \dots$. Let

$$B = \left\{ \lim_{k \rightarrow \infty} p_{nm}(s'_k - t) \right\}.$$

We note first that $B \in \mathfrak{M}$, and we also note that

$$(2.16) \quad \sum_r b_{nr} p_{rm}(t) \leq \sum_r p_{nr}(t) b_{rm} = a_{nm}.$$

Summing both sides of (2.16) over m , we obtain

$$(2.17) \quad \sum_r b_{nr} \leq \sum_r a_{nr}.$$

Now, noting that only equality is possible in (2.17), since $A \in \mathfrak{N}$, we see, by (2.16), that

$$(2.18) \quad B \cdot P(t) = P(t) \cdot B = A.$$

Multiplying (2.18) by $P(t)$ yields

$$(2.19) \quad A \cdot P(t) = P(t) \cdot B \cdot P(t) = P(t) \cdot A.$$

By (i), $P(t) \cdot A \in \mathfrak{M}$. Summing over the i th row of $P(t) \cdot A = A \cdot P(t)$, yields $\sum_r a_{ir}$. Hence $P(t) \cdot A \in \mathfrak{N}$, since $A \in \mathfrak{N}$. This concludes the proof of (ii).

We may now show that there exists a matrix A' as above. Indeed, let B be any matrix in \mathfrak{N} ; and $\{t_k\} \uparrow \infty$ a sequence such that $\{s_{u_k} - t_k\} \uparrow \infty$, and such that

$$B = \{b_{nm}\} = \left\{ \lim_{k \rightarrow \infty} p_{nm}(s_{u_k} - t_k) \right\}$$

for some subsequence $\{s_{u_k}\}$ of $\{s_k\}$, where $\{s_k\}$ is the sequence of (ii). Let $\{t_k\}$ have the further property that $\lim_{k \rightarrow \infty} p_{nm}(t_k) = c_{nm}$ exists for $n, m = 1, 2, \dots$. By (1.3) we have that

$$p_{nm}(s_{u_k}) = \sum_r p_{nr}(s_{u_k} - t_k) p_{rm}(t_k),$$

and hence that there exist matrices $B \in \mathfrak{N}$ and $C \in \mathfrak{Q}$ such that $A \geq BC$. Since

$$\lim_{k \rightarrow \infty} \sum_r b_{nr} p_{rm}(t_k) = \sum_r b_{nr} c_{rm},$$

$B \cdot C$ is the limit matrix of $B \cdot P(t_k)$ as $k \rightarrow \infty$. By (ii), since $B \in \mathfrak{N}$, $B \cdot P(t_k) \in \mathfrak{N}$, and since \mathfrak{N} is closed, the limit matrix $B \cdot C$ is also in \mathfrak{N} . Finally, take $A' = B \cdot C$.

We have shown that given any limiting matrix A there exists a matrix $A' \in \mathfrak{M}$ such that $A \geq A'$. In particular, if A is such that

$$a_{jj} = \liminf_{t \rightarrow \infty} p_{jj}(t),$$

then,

$$\liminf_{t \rightarrow \infty} p_{jj}(t) \geq \limsup_{t \rightarrow \infty} p_{jj}(t),$$

or $\lim_{t \rightarrow \infty} p_{jj}(t)$ exists. Applying Corollary 1 completes the proof.

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