CHARACTERISTIC FUNCTIONALS
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Let $E$ be a partially ordered linear normed space over the field of real numbers whose positive cone $K$ has nonempty interior $K^0$. A functional $f$ in $E^*$, the conjugate space of $E$, is called positive if $f(K) \geq 0$ and $f \neq 0$. A semi-group $\Gamma$ of linear operators on $E$ is called positive if $\Gamma(K) \subseteq K$. It will be assumed that $\Gamma$ contains the identity operator.

This note is concerned with the properties of positive semi-groups $\Gamma$ which guarantee the existence of characteristic (invariant) positive functionals relative to $\Gamma$, i.e., $A*f = \lambda_A f (A*f = f), \lambda_A > 0, A \in \Gamma$. It will be assumed throughout this paper that

(*) $\Gamma(u) \subseteq K^0$ for some $u \in K^0$.

This note has as its starting point a theorem of Krein and Rutman [4, p. 40, Theorem 3.3]:

**Theorem 1 (Krein-Rutman).** If $\Gamma$ is commutative, then $\Gamma$ has characteristic positive functions.

**Proof.** The gist of the proof of this theorem consists of showing that (a) each $A \in \Gamma$ has a set $H_A$ of characteristic positive functionals and (b) the intersection of all $H_A (A \in \Gamma)$ is not empty. The Krein-Rutman proof, with a minor correction, is briefly summarized below.

Let $H$ be the set of positive functionals $f$ satisfying the equation $f(u) = 1$. $H$ is a $w^*$-compact convex subset of the conjugate space $E^*$. To each $A \in \Gamma$ one associates a continuous map $B: H \rightarrow H$ defined by the equation

$$Bf = A*f/f(Au) \quad (A \in \Gamma, f \in H).$$

Since $H$ is $w^*$-compact and convex, $B$ has a fixed point, by the Schauder fixed point theorem. Let $H_A$ denote the set of fixed points of $B$, then $A*f = \lambda_A f (f \in H_A)$. Let $B'$ be the continuous map associated with an $A' \in \Gamma$. Then

(2) $A*B'f = A*A'*f/f(A'u) = A'*A*f/f(A'u) = \lambda_A A'*f/f(A'u) = \lambda_A B'f \quad (f \in H_A)$.

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1 The ordering is assumed to be nontrivial, i.e. $0 \notin K^0$.

2 This condition is equivalent to $\Gamma(K^0) \subseteq K^0$. 

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Thus $B'(H_A) \subset H_A$. Moreover, $B'$ preserves the characteristic value $\lambda_{Af}$. Hence, if $K_A$ is a subset of $H_A$ consisting of all functionals $f$ having equal $\lambda_{Af}$, $B'(K_A) \subset K_A$. As $K_A$ is $\omega^*$-compact and convex, $B'$ has a fixed point in $K_A$. Therefore the set $\{H_A \mid A \in \Gamma\}$ has the finite intersection property, and $\bigcap_{A \in \Gamma} H_A$ is nonempty.

Kreln and Rutman made the mistake of asserting that $H_A$ was convex. If this were true, one would have an extension of Theorem 1 to left solvable semi-groups of linear operators which would be false. (The proof would follow as in the proof of Theorem 2 below.) A subsemi-group $\Gamma^{(1)}$ of $\Gamma$ is called a left commutator subsemi-group if for every $A, A'$ in $\Gamma$ there exists an $A''$ in $\Gamma^{(1)}$ such that $AA' = A''A'A$. A semi-group $\Gamma$ is called left solvable if there exists a sequence of subsemi-groups $\Gamma = \Gamma^{(0)} \supset \Gamma^{(1)} \supset \cdots \supset \Gamma^{(n)}$ such that (a) $\Gamma^{(n)}$ is commutative and (b) each $\Gamma^{(i)}$ is a left commutator subsemi-group of $\Gamma^{(i-1)}$ ($i = 1, \cdots, n$).

Consider, for example, the Banach space $E$ of the euclidean plane and the commutative group $G^{(1)}$ of $2 \times 2$ matrices

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad (a > 0, b > 0).$$

Let $K = \{(x, y) \mid x \geq 0, y \geq 0\}$ and $u = (1, 1)$. Then, unless $A$ is a scalar matrix, $H_A$ consists of only two elements $f, g$ where $f((x, y)) = x$ and $g((x, y)) = y$. If one adjoins to $G^{(1)}$ the matrix

$$A_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

one obtains a solvable group $G$ whose commutator subgroup is $G^{(1)}$. Since $A_0^*f = g$, $A_0^*g = f$, $G$ has no characteristic functionals.

The particular role played by the commutativity of $\Gamma$ lies in the fact that $B'(K_A) \subset K_A$. This suggests the following extension of Theorem 1.

**Theorem 2.** Let $\Gamma^{(1)}$ be a left commutator subsemi-group of $\Gamma$. If $\Gamma^{(1)}$ has an invariant positive functional, then $\Gamma$ has a characteristic positive functional.

**Proof.** Let $II$ be the $\omega^*$-compact convex set of invariant positive functionals $f$ relative to $\Gamma^{(1)}$ such that $f(u) = 1$. Let $B$ be the continuous map associated with an element $A \in \Gamma$ as defined by the equation (1). For any $A' \in \Gamma^{(1)}$ there exists $A'' \in \Gamma^{(1)}$ such that $AA' = A''A'A$. Then, for $f \in H$,

Hence \( B(H) \subset H \) and, consequently, \( A \) has a characteristic positive functional in \( H \). Let \( K_A \) be a convex subset of \( H \) consisting of all characteristic functionals of \( A \) having a common characteristic value \( \lambda_A \). If \( B_1 \) is the continuous map associated with an element \( A_1 \in \Gamma \) and if \( f \in K_A \), then, since \( A_1A = A_2AA_1 \) \((A_2 \in \Gamma^{(1)})\),

\[ A^*B_1f = A^*A_1^*f/f(A_1u) = A_1^*A^*A_2^*f/f(A_1u) = A_1^*A^*f/f(A_1u) = \lambda_A B_1f. \]

Thus \( B_1(K_A) \subset K_A \) and \( A_1 \) has a characteristic functional in \( K_A \). Consequently, \( \bigcap_{A \in \Gamma} K_A \) is not empty.

**Theorem 3.** Suppose that \( \Gamma \), considered as an abstract semi-group, has a right invariant mean. Then the following statements are equivalent.

(a) \( \Gamma \) has a characteristic positive functional.

(b) There exists a closed hyperplane \( X \subset E \) such that \( X \cap K^0 = \emptyset \) and \( \Gamma(X) \subset X \).

(c) There exists a positive functional \( f \) such that

\[ f(AA'u) = f(Au)f(A'u) \quad (A, A' \in \Gamma). \]

(d) There exists a positive functional \( f \) such that

\[ f(AA'A''u) = f(A'^*AA''u) \quad (A, A', A'' \in \Gamma). \]

(e) There exist a homomorphism \( \lambda: A \rightarrow \lambda_A \) of \( \Gamma \) into the multiplicative group of the positive real numbers, two positive real numbers \( m \) and \( M \), and a positive functional \( f \) such that

\[ m\lambda_A \leq f(Au) \leq M\lambda_A, \quad A \in \Gamma. \]

**Proof.** In order to clarify the role of the right invariant mean, one first proves without any assumption on \( \Gamma \) that (a) is equivalent to (b), (c) is equivalent to (d), and that (a) implies (c) and (c) implies (e).

(a) \( \Rightarrow \) (b). If \( f \) is a characteristic positive functional relative to \( \Gamma \), then \( X = \{ x | f(x) = 0 \} \) satisfies the requirements of (b).

(b) \( \Rightarrow \) (a). Let \( f \) be a functional such that \( X = f^{-1}(0) \) and \( f(u) = 1 \). If \( y = tu + x \geq 0 \) \((x \in X, t \) a real number\) then \( x = y - tu \geq -tu \). Since \( X \cap K^0 = \emptyset \), \( t \geq 0 \) and \( f \) is positive. For every \( y \in E \), \( y - f(y)u \in X \).

\(^3\) Let \( m(\Gamma) \) denote the Banach space of bounded real-valued functions on \( \Gamma \). A positive functional \( \mu \) in \( m(\Gamma)^* \) is called a right invariant mean if \( \mu(x) = \mu(R_{Ax})(x \in m(\Gamma), A \in \Gamma) \) and \( \mu(I) = 1 \). where \( R_{Ax}(A') = x(A'A) \) and \( I \) is the constant 1 function.
Therefore $Ay-f(y)Au \subseteq X$ for any $A \in \Gamma$. Then $f(Ay) = f(Au)f(y)$ and $f$ is a characteristic positive functional.

(c) $\Rightarrow$ (d). This is clear.

(d) $\Rightarrow$ (c). Let $U$ be the linear subspace generated by $\{Au | A \in \Gamma\}$. Since $U$ contains $u$, any positive functional in $U^*$ can be extended to a positive functional in $E^*$ [4, Theorem 1.1, p. 13]. If a positive functional $f \in U^*$ is characteristic relative to $\Gamma$ and if $f(u) = 1$ then $f$ satisfies (c), for $f(Au) = \lambda_A$. This proves (a) $\Rightarrow$ (c). Let $H$ be the set of positive functionals $f \in U^*$ which satisfies the condition (d) and the condition $f(u) = 1$. Restricted to $H$, $\Gamma$ is commutative. From this point one uses the argument of the proof of Theorem 1 to produce a characteristic functional.

(a) $\Rightarrow$ (c) $\Rightarrow$ (e). Any characteristic functional $f$ with $f(u) = 1$ satisfies (c). If $f$ satisfies (c), then $f$ satisfies (e) with $\lambda_A = f(Au)$ and $m = M = 1$.

Finally one shows (e) $\Rightarrow$ (a) with the added hypothesis that $\Gamma$ has a right invariant mean. The map $A \rightarrow \lambda_A^{-1}A$ is a homomorphism of $\Gamma$ onto a semi-group $\Gamma'$ of positive linear operators on $E$. $\Gamma'$, a homomorphic image of $\Gamma$, also has a right invariant mean, and $\Gamma'(u) \subset K^0$. With respect to $\Gamma'$ one has the relation $m \leq f(A'u) \leq M (A' \in \Gamma')$. If a positive functional $f$ is invariant under $\Gamma'$ then $f$ is a characteristic functional relative to $\Gamma$, in fact, $\lambda_{A'} = \lambda_A$. Hence for the validity of (e) $\Rightarrow$ (a) it is sufficient to prove:

**Lemma 1.** If $\Gamma$ has a right invariant mean and if there exist a positive functional $f$ and positive real numbers $m, M$ such that

$$0 < m \leq f(Au) \leq M \quad (A \in \Gamma),$$

then $\Gamma$ has an invariant positive functional.

**Proof.** Since $u \in K^0$, there is a sphere of radius $r > 0$ about $u$ which is contained in $K$. Therefore if $\|x\| < r, \ u \pm x \geq 0$. Hence $\|f(Ax)\| \leq f(Au) \leq M \ (A \in \Gamma)$. In general, $|f(Ax)| \leq M\|x\|/r$. Then, for each $x \in E$, the map $F(x) : A \rightarrow f(Ax) \ (A \in \Gamma')$ is a bounded function on $\Gamma'$. Clearly, $F(ax+by) = aF(x) + bF(y)$ for any $x, y \in E$ and real numbers $a, b$. Let $\mu$ be a right invariant mean of $\Gamma$. Since, for fixed $A_0$,

*Sketch of proof: The homomorphism $A \rightarrow A'$ of $\Gamma$ onto $\Gamma'$ induces a homomorphism $x' \rightarrow x$ of $m(\Gamma')$ into $m(\Gamma)$, where $x(A) = x'(A')$. Then the image of $R_Bx'$ is $R_Bx$ where $B \in \Gamma$ is an inverse image of $B' \in \Gamma'$. Through this identification, the right invariant mean of $\Gamma$ induces a right invariant mean of $\Gamma'$. Since any representation of a homomorphic image is also a representation of the original semi-group, one may also follow [2, Theorem 2, p. 280] to obtain a proof. See also, M. M. Day, *Amenable semigroups*, Illinois J. of Math. vol. 1 (1957), pp. 509–544.*
$F(x)(AA_0) = f(AA_0 x) = F(A_0 x)(A)$, $\mu F(x) = \mu F(A_0 x)$. Thus the functional $\phi$ defined by $\phi: x \mapsto \mu F(x)$ is invariant under $\Gamma$. $\phi$ is positive. Moreover, $\phi \neq 0$, for $\phi(u) = \mu F(u) \geq m$. This completes the proof.

Remark. The proof that (c), (d) imply (a) may in essence be considered as the problem of extending a positive functional $f$ in $U^*$ invariant relative to the operators $A^*/f(Au)$ to a positive invariant functional in $E^*$. It is in this respect that the hypothesis of a right invariant mean is used. In the proof of the lemma, if $f(Ax_0) = f(x_0)$ ($A \in \Gamma$) for some $x_0$, then $f(x_0) = \phi(x_0)$. The conditions (c), (d) do not by themselves imply the existence of a positive characteristic functional. Consider for example a semi-group $G$ which does not have invariant mean (e.g. a free group on two generators [2, p. 290]). Let $m(G)$ be the Banach lattice of bounded real-valued functions on $G$, and $G^*$ the right regular representation of $G$ on $m(G)$. Let $u$ be the constant one function in $m(G)$. Then $u$ is invariant under $G^*$ and (d) is satisfied. Since $u = Au$ ($A \in G^*$), any positive characteristic functional is a right invariant mean. Therefore, there is no positive characteristic functional relative to $G^*$.

Corollary 1. If $\Gamma$ is a locally finite group then $\Gamma$ has an invariant positive functional.

Proof. Let $\Gamma_\delta$ denote the finite subgroup generated by a finite subset $\delta$ of $\Gamma$. Then $\Gamma_\delta$ has invariant means and satisfies (4). Hence each $\Gamma_\delta$ has a set $H_\delta$ of invariant positive functionals $f$ with $f(u) = 1$. If $\delta \supseteq \delta'$ then $H_\delta \subseteq H_{\delta'}$. Hence $\cap_\delta H_\delta$ is not empty and $\Gamma$ has an invariant positive functional.

Corollary 1 is not true for locally finite semi-groups, for there are finite semi-groups having no invariant mean. But if a semigroup is a set theoretical union of a directed (by set inclusion) set of subsemi-groups and if each of these subsemi-groups has invariant functionals, then the semi-group has an invariant functional.

As another consequence of Lemma 1 is a theorem of Civin and Yood [1, Theorem 4.1], which is stated in a slightly different form as follows:

Corollary 2. Let $\Gamma$ be a left solvable semi-group, $\Gamma_1$ the convex combination of $\Gamma$. Suppose that

(i) for some $s > 0$, $Au \geq su$, $A \in \Gamma$, and

(ii) for some $r \geq 0$, given $A \in \Gamma$ there exists $A_1 \in \Gamma_1$ such that $A_1 u \leq ru$ and $A_1 Au \leq ru$. Then $\Gamma$ has an invariant positive functional.

Proof. A left solvable semi-group has a right invariant mean. This fact is probably well-known; but, unfortunately, we do not have
an explicit reference. One may obtain a proof from [2, Theorem 5] by properly separating "left" and "right." We shall indicate a direct proof at the end of this note, it is analogous to the proof of [2, Theorem 6].

Next one shows that there exist positive real numbers \( m \) and \( M \) such that

\[
(5) \quad mu \leq Au \leq Mu \quad (A \in \Gamma).
\]

Take \( m = s \) for the lower bound. Note that (i) also holds for elements of \( \Gamma_1 \). Suppose that for any positive integer \( n \) there exists \( A \in \Gamma \) such that \( Au \geq nu \). Choose \( A_1 \in \Gamma_1 \) to match \( A \) according to (ii). Then \( ru \geq A_1Au \geq A_1nu \geq nsu \). Since \( n \) is arbitrary, \( s = 0 \). This contradicts (i). Therefore (5) holds, hence (4) is satisfied. Thus by Lemma 1, \( \Gamma \) has an invariant positive functional.

Although all our results deal only with semi-groups having invariant means, this is not a necessary condition. For example, let \( f \) be a positive functional. Then \( E \) is a direct sum of the one-dimensional subspace generated by \( u \) and the subspace \( f^{-1}(0) \). Then any set of positive linear operators which leaves \( u \) and \( f^{-1}(0) \) invariant has \( f \) as an invariant functional. If \( E \) is a 4-dimensional space then \( f^{-1}(0) = E_3 \) is a euclidean 3-space. The group of all rotations in \( E_3 \) which contains a free group on two generators, has no invariant mean.

To conclude this note we indicate the proof that a left solvable semi-group has a right invariant mean. Let \( S \) be a left solvable semi-group and let \( S' \) be a left commutator subsemi-group of \( S \). By induction one may assume that \( S' \) has a right invariant mean. For \( A, B \in S \), define \( A \sim B \) if and only if there exist \( A', B' \in S' \) such that \( A'A = B'B \), [3]. (One or both of \( A', B' \) may be the identity which need not be in \( S \).) Define \( A \equiv B \) (\( A \) is equivalent to \( B \)) if there exist \( A = A_1, A_2, \ldots , A_n = B \) such that \( A_i \sim A_{i+1}, 1 \leq i \leq n - 1 \). One now proves that (1) if \( A_1 \sim B_1 \) and \( A_2 \sim B_2 \), then \( A_1A_2 \sim B_1B_2 \), (2) \( AB \sim BA \), (3) if \( A_1 \equiv B_1 \) and \( A_2 \equiv B_2 \), then \( A_1A_2 \equiv B_1B_2 \), and (4) \( AB \equiv BA \). Let \( \overline{A} \) denote the equivalence class of \( A \) and define \( \overline{A} \cdot \overline{B} = \overline{AB} \). Then, by (3) and (4), the set \( \overline{S} \) of equivalence classes forms a commutative semi-group. Therefore \( \overline{S} \) has an invariant mean.

Let \( m(S) \) be the space of bounded real-valued functions on \( S \). If \( x \in m(S) \) and \( A \in S \) let \( x_A \) be the function in \( m(S') \) defined by \( x_A(A') = x(A'A) \) \( (A' \in S') \). Let \( \nu \) be a right invariant mean of \( S' \) and \( \chi \) an invariant mean of \( \overline{S} \). Then, for a fixed \( x \in m(S) \), the function \( \nu(x_A) \) \( (A \in S) \) is constant on \( \overline{A} \). Therefore \( \hat{x} \) defined by \( \hat{x}(\overline{A}) = \nu(x_A) \) is in \( m(\overline{S}) \). Then \( \mu(x) = \chi(\hat{x}) \) is a right invariant mean of \( S \).


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**QUASI-NIL RINGS**

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Rings have been studied which have among others the property that every commutator $xy - yx$ is in the nucleus. It seems appropriate to consider rings in which the square of every element is in the nucleus, a property that is shared by both associative and Lie rings. Under the additional assumptions of primeness and characteristic different from 2 it can be shown that such rings are either associative or have the property that $x^2 = 0$, for every element $x$ of the ring. If further $(x, y, z) + (y, z, x) + (z, x, y)$ is in the nucleus for all elements $x, y, z$ of the ring, then the ring is either associative or a Lie ring.

We use the notation $(x, y, z) = (xy)z - x(yz)$. The nucleus $N$ of a ring $R$ consists of all $n \in R$ such that $(n, R, R) = (R, n, R) = (R, R, n) = 0$. $N$ is a subring of $R$.

**Lemma.** Let $R$ be a prime ring satisfying $x^2 \in N$ for every $x \in R$ and of characteristic different from 2. Then either $R$ is associative or $N^2 = 0$.

**Proof.** For all $r, s \in R$, $rs + sr = (r + s)^2 - r^2 - s^2$ must be in $N$. Select $n, n' \in N$, and $x, y, z \in R$. Then $(n(n'x + xn'), y, z) = 0$, so that $(n'n'x, y, z) = -(nxn', y, z)$. Similarly $(nxn', y, z) = -(n'n'x, y, z)$ and $(xn'n', y, z) = -(n'n'x, y, z)$. By combining these three equalities it follows that $2(n'n'x, y, z) = 0$. Assuming characteristic not 2 it then follows that $(n'n'x, y, z) = 0$. Since $(nx, y, z) = ((nx)y)z - (nx)(yz) = (n(xy))z - n(x(yz)) = n((xy)z) - n(x(yz)) = n(x, y, z)$, we replace $n$

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