

CHARACTERISTIC FUNCTIONALS

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Let E be a partially ordered linear normed space over the field of real numbers whose positive cone K has nonempty interior K^0 .¹ A functional f in E^* , the conjugate space of E , is called *positive* if $f(K) \geq 0$ and $f \neq 0$. A semi-group Γ of linear operators on E is called *positive* if $\Gamma(K) \subset K$. It will be assumed that Γ contains the identity operator.

This note is concerned with the properties of positive semi-groups Γ which guarantee the existence of characteristic (invariant) positive functionals relative to Γ , i.e., $A^*f = \lambda_A f$ ($A^*f = f$), $\lambda_A > 0$, $A \in \Gamma$. It will be assumed throughout this paper that²

$$(*) \quad \Gamma(u) \subset K^0 \quad \text{for some } u \in K^0.$$

This note has as its starting point a theorem of Kreĭn and Rutman [4, p. 40, Theorem 3.3]:

THEOREM 1 (KREĬN-RUTMAN). *If Γ is commutative, then Γ has characteristic positive functions.*

PROOF. The gist of the proof of this theorem consists of showing that (a) each $A \in \Gamma$ has a set H_A of characteristic positive functionals and (b) the intersection of all H_A ($A \in \Gamma$) is not empty. The Kreĭn-Rutman proof, with a minor correction, is briefly summarized below.

Let H be the set of positive functionals f satisfying the equation $f(u) = 1$. H is a w^* -compact convex subset of the conjugate space E^* . To each $A \in \Gamma$ one associates a continuous map $B: H \rightarrow H$ defined by the equation

$$(1) \quad Bf = A^*f/f(Au) \quad (A \in \Gamma, f \in H).$$

Since H is w^* -compact and convex, B has a fixed point, by the Schauder fixed point theorem. Let H_A denote the set of fixed points of B , then $A^*f = \lambda_A f$ ($f \in H_A$). Let B' be the continuous map associated with an $A' \in \Gamma$. Then

$$(2) \quad \begin{aligned} A^*B'f &= A^*A'^*f/f(A'u) = A'^*A^*f/f(A'u) = \lambda_{A'}A^*f/f(A'u) \\ &= \lambda_{A'}B'f \end{aligned} \quad (f \in H_A).$$

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¹ The ordering is assumed to be nontrivial, i.e. $0 \notin K^0$.

² This condition is equivalent to $\Gamma(K^0) \subset K^0$.

Thus $B'(H_A) \subset H_A$. Moreover, B' preserves the characteristic value λ_{Af} . Hence, if K_A is a subset of H_A consisting of all functionals f having equal λ_{Af} , $B'(K_A) \subset K_A$. As K_A is w^* -compact and convex, B' has a fixed point in K_A . Therefore the set $\{H_A \mid A \in \Gamma\}$ has the finite intersection property, and $\bigcap_{A \in \Gamma} H_A$ is nonempty.

Kreĭn and Rutman made the mistake of asserting that H_A was convex. If this were true, one would have an extension of Theorem 1 to left solvable semi-groups of linear operators which would be false. (The proof would follow as in the proof of Theorem 2 below.) A subsemi-group $\Gamma^{(1)}$ of Γ is called a *left commutator subsemi-group* if for every A, A' in Γ there exists an A'' in $\Gamma^{(1)}$ such that $AA' = A''A'A$. A semi-group Γ is called left solvable if there exists a sequence of subsemi-groups $\Gamma = \Gamma^{(0)} \supset \Gamma^{(1)} \supset \dots \supset \Gamma^{(n)}$ such that (a) $\Gamma^{(n)}$ is commutative and (b) each $\Gamma^{(i)}$ is a left commutator subsemi-group of $\Gamma^{(i-1)}$ ($i = 1, \dots, n$).

Consider, for example, the Banach space E of the euclidean plane and the commutative group $G^{(1)}$ of 2×2 matrices

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad (a > 0, b > 0).$$

Let $K = \{(x, y) \mid x \geq 0, y \geq 0\}$ and $u = (1, 1)$. Then, unless A is a scalar matrix, H_A consists of only two elements f, g where $f((x, y)) = x$ and $g((x, y)) = y$. If one adjoins to $G^{(1)}$ the matrix

$$A_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

one obtains a solvable group G whose commutator subgroup is $G^{(1)}$. Since $A_0^*f = g, A_0^*g = f$, G has no characteristic functionals.

The particular role played by the commutativity of Γ lies in the fact that $B'(K_A) \subset K_A$. This suggests the following extension of Theorem 1.

THEOREM 2. *Let $\Gamma^{(1)}$ be a left commutator subsemi-group of Γ . If $\Gamma^{(1)}$ has an invariant positive functional, then Γ has a characteristic positive functional.*

PROOF. Let H be the w^* -compact convex set of invariant positive functionals f relative to $\Gamma^{(1)}$ such that $f(u) = 1$. Let B be the continuous map associated with an element $A \in \Gamma$ as defined by the equation (1). For any $A' \in \Gamma^{(1)}$ there exists $A'' \in \Gamma^{(1)}$ such that $AA' = A''A'A$. Then, for $f \in H$,

$$A^*Bf = A^*A^*f/f(Au) = A^*A^*A''^*f/f(Au) = A^*f/f(Au) = Bf.$$

Hence $B(H) \subset H$ and, consequently, A has a characteristic positive functional in H . Let K_A be a convex subset of H consisting of all characteristic functionals of A having a common characteristic value λ_{A_f} . If B_1 is the continuous map associated with an element $A_1 \in \Gamma$ and if $f \in K_A$, then, since $A_1A = A_2AA_1$ ($A_2 \in \Gamma^{(1)}$),

$$\begin{aligned} A^*B_1f &= A^*A_1^*f/f(A_1u) = A_1^*A^*A_2^*f/f(A_1u) = A_1^*A^*f/f(A_1u) \\ &= \lambda_{A_f}B_1f. \end{aligned}$$

Thus $B_1(K_A) \subset K_A$ and A_1 has a characteristic functional in K_A . Consequently, $\bigcap_{A \in \Gamma} K_A$ is not empty.

THEOREM 3. *Suppose that Γ , considered as an abstract semi-group, has a right invariant mean.³ Then the following statements are equivalent.*

(a) Γ has a characteristic positive functional.
 (b) There exists a closed hyperplane $X \subset E$ such that $X \cap K^0 = \emptyset$ and $\Gamma(X) \subset X$.

(c) There exists a positive functional f such that

$$f(AA'u) = f(Au)f(A'u) \quad (A, A' \in \Gamma).$$

(d) There exists a positive functional f such that

$$f(AA'A''u) = f(A'A''u) \quad (A, A', A'' \in \Gamma).$$

(e) There exist a homomorphism $\lambda: A \rightarrow \lambda_A$ of Γ into the multiplicative group of the positive real numbers, two positive real numbers m and M , and a positive functional f such that

$$(3) \quad m\lambda_A \leq f(Au) \leq M\lambda_A, \quad A \in \Gamma.$$

PROOF. In order to clarify the role of the right invariant mean, one first proves without any assumption on Γ that (a) is equivalent to (b), (c) is equivalent to (d), and that (a) implies (c) and (c) implies (e).

(a) \Rightarrow (b). If f is a characteristic positive functional relative to Γ , then $X = \{x | f(x) = 0\}$ satisfies the requirements of (b).

(b) \Rightarrow (a). Let f be a functional such that $X = f^{-1}(0)$ and $f(u) = 1$. If $y = tu + x \geq 0$ ($x \in X$, t a real number) then $x = y - tu \geq -tu$. Since $X \cap K^0 = \emptyset$, $t \geq 0$ and f is positive. For every $y \in E$, $y - f(y)u \in X$.

³ Let $m(\Gamma)$ denote the Banach space of bounded real-valued functions on Γ . A positive functional μ in $m(\Gamma)^*$ is called a right invariant mean if $\mu(x) = \mu(R_Ax)$ ($x \in m(\Gamma)$, $A \in \Gamma$) and $\mu(I) = 1$, where $R_Ax(A') = x(A'A)$ and I is the constant 1 function.

Therefore $Ay - f(y)Au \in X$ for any $A \in \Gamma$. Then $f(Ay) = f(Au)f(y)$ and f is a characteristic positive functional.

(c) \Rightarrow (d). This is clear.

(d) \Rightarrow (c). Let U be the linear subspace generated by $\{Au \mid A \in \Gamma\}$. Since U contains u , any positive functional in U^* can be extended to a positive functional in E^* [4, Theorem 1.1, p. 13]. If a positive functional $f \in U^*$ is characteristic relative to Γ and if $f(u) = 1$ then f satisfies (c), for $f(Au) = \lambda_{Af}$. (This proves (a) \Rightarrow (c).) Let H be the set of positive functionals $f \in U^*$ which satisfies the condition (d) and the condition $f(u) = 1$. Restricted to H , Γ is commutative. From this point one uses the argument of the proof of Theorem 1 to produce a characteristic functional.

(a) \Rightarrow (c) \Rightarrow (e). Any characteristic functional f with $f(u) = 1$ satisfies (c). If f satisfies (c), then f satisfies (e) with $\lambda_A = f(Au)$ and $m = M = 1$.

Finally one shows (e) \Rightarrow (a) with the added hypothesis that Γ has a right invariant mean. The map $A \rightarrow \lambda_A^{-1}A$ is a homomorphism of Γ onto a semi-group Γ' of positive linear operators on E . Γ' , a homomorphic image of Γ , also has a right invariant mean,⁴ and $\Gamma'(u) \subset K^0$. With respect to Γ' one has the relation $m \leq f(A'u) \leq M$ ($A' \in \Gamma'$). If a positive functional f is invariant under Γ' then f is a characteristic functional relative to Γ , in fact, $\lambda_{Af} = \lambda_A$. Hence for the validity of (e) \Rightarrow (a) it is sufficient to prove:

LEMMA 1. *If Γ has a right invariant mean and if there exist a positive functional f and positive real numbers m, M such that*

$$(4) \qquad 0 < m \leq f(Au) \leq M \qquad (A \in \Gamma),$$

then Γ has an invariant positive functional.

PROOF. Since $u \in K^0$, there is a sphere of radius $r > 0$ about u which is contained in K . Therefore if $\|x\| < r$, $u \pm x \geq 0$. Hence $|f(Ax)| \leq f(Au) \leq M$ ($A \in \Gamma$). In general, $|f(Ax)| \leq M\|x\|/r$. Then, for each $x \in E$, the map $F(x): A \rightarrow f(Ax)$ ($A \in \Gamma$) is a bounded function on Γ . Clearly, $F(ax + by) = aF(x) + bF(y)$ for any $x, y \in E$ and real numbers a, b . Let μ be a right invariant mean of Γ . Since, for fixed A_0 ,

⁴ Sketch of proof: The homomorphism $A \rightarrow A'$ of Γ onto Γ' induces a homomorphism $x' \rightarrow x$ of $m(\Gamma')$ into $m(\Gamma)$, where $x(A) = x'(A')$. Then the image of $R_{B'}x'$ is R_Bx where $B \in \Gamma$ is an inverse image of $B' \in \Gamma'$. Through this identification, the right invariant mean of Γ induces a right invariant mean of Γ' . Since any representation of a homomorphic image is also a representation of the original semi-group, one may also follow [2, Theorem 2, p. 280] to obtain a proof. See also, M. M. Day, *Amenable semigroups*, Illinois J. of Math. vol. 1 (1957), pp. 509-544.

$F(x)(AA_0) = f(AA_0x) = F(A_0x)(A)$, $\mu F(x) = \mu F(A_0x)$. Thus the functional ϕ defined by $\phi: x \rightarrow \mu F(x)$ is invariant under Γ . ϕ is positive. Moreover, $\phi \neq 0$, for $\phi(u) = \mu F(u) \geq m$. This completes the proof.

REMARK. The proof that (c), (d) imply (a) may in essence be considered as the problem of extending a positive functional f in U^* invariant relative to the operators $A^*/f(Au)$ to a positive invariant functional in E^* . It is in this respect that the hypothesis of a right invariant mean is used. In the proof of the lemma, if $f(Ax_0) = f(x_0)$ ($A \in \Gamma$) for some x_0 , then $f(x_0) = \phi(x_0)$. The conditions (c), (d) do not by themselves imply the existence of a positive characteristic functional. Consider for example a semi-group G which does not have invariant mean (e.g. a free group on two generators [2, p. 290]). Let $m(G)$ be the Banach lattice of bounded real-valued functions on G , and G^* the right regular representation of G on $m(G)$. Let u be the constant one function in $m(G)$. Then u is invariant under G^* and (d) is satisfied. Since $u = Au$ ($A \in G^*$), any positive characteristic functional is a right invariant mean. Therefore, there is no positive characteristic functional relative to G^* .

COROLLARY 1. *If Γ is a locally finite group then Γ has an invariant positive functional.*

PROOF. Let Γ_δ denote the finite subgroup generated by a finite subset δ of Γ . Then Γ_δ has invariant means and satisfies (4). Hence each Γ_δ has a set H_δ of invariant positive functionals f with $f(u) = 1$. If $\delta \supseteq \delta'$ then $H_\delta \subseteq H_{\delta'}$. Hence $\bigcap_\delta H_\delta$ is not empty and Γ has an invariant positive functional.

Corollary 1 is not true for locally finite semi-groups, for there are finite semi-groups having no invariant mean. But if a semigroup is a set theoretical union of a directed (by set inclusion) set of subsemi-groups and if each of these subsemi-groups has invariant functionals, then the semi-group has an invariant functional.

As another consequence of Lemma 1 is a theorem of Civin and Yood [1, Theorem 4.1], which is stated in a slightly different form as follows:

COROLLARY 2. *Let Γ be a left solvable semi-group, Γ_1 the convex combination of Γ . Suppose that*

- (i) *for some $s > 0$, $Au \geq su$, $A \in \Gamma$, and*
- (ii) *for some $r \geq 0$, given $A \in \Gamma$ there exists $A_1 \in \Gamma_1$ such that $A_1u \leq ru$ and $A_1Au \leq ru$. Then Γ has an invariant positive functional.*

PROOF. A left solvable semi-group has a right invariant mean. This fact is probably well-known; but, unfortunately, we do not have

an explicit reference. One may obtain a proof from [2, Theorem 5] by properly separating “left” and “right.” We shall indicate a direct proof at the end of this note, it is analogous to the proof of [2, Theorem 6].

Next one shows that there exist positive real numbers m and M such that

$$(5) \quad mu \leq Au \leq Mu \quad (A \in \Gamma).$$

Take $m = s$ for the lower bound. Note that (i) also holds for elements of Γ_1 . Suppose that for any positive integer n there exists $A \in \Gamma$ such that $Au \geq nu$. Choose $A_1 \in \Gamma_1$ to match A according to (ii). Then $ru \geq A_1Au \geq A_1nu \geq nsu$. Since n is arbitrary, $s = 0$. This contradicts (i). Therefore (5) holds, hence (4) is satisfied. Thus by Lemma 1, Γ has an invariant positive functional.

Although all our results deal only with semi-groups having invariant means, this is not a necessary condition. For example, let f be a positive functional. Then E is a direct sum of the one-dimensional subspace generated by u and the subspace $f^{-1}(0)$. Then any set of positive linear operators which leaves u and $f^{-1}(0)$ invariant has f as an invariant functional. If E is a 4-dimensional space then $f^{-1}(0) = E_3$ is a euclidean 3-space. The group of all rotations in E_3 which contains a free group on two generators, has no invariant mean.

To conclude this note we indicate the proof that a left solvable semi-group has a right invariant mean. Let S be a left solvable semi-group and let S' be a left commutator subsemi-group of S . By induction one may assume that S' has a right invariant mean. For $A, B \in S$, define $A \sim B$ if and only if there exist $A', B' \in S'$ such that $A'A = B'B$, [3]. (One or both of A', B' may be the identity which need not be in S .) Define $A \approx B$ (A is equivalent to B) if there exist $A = A_1, A_2, \dots, A_n = B$ such that $A_i \sim A_{i+1}, 1 \leq i \leq n-1$. One now proves that (1) if $A_1 \sim B_1$ and $A_2 \sim B_2$, then $A_1A_2 \sim B_1B_2$, (2) $AB \sim BA$, (3) if $A_1 \approx B_1$ and $A_2 \approx B_2$, then $A_1A_2 \approx B_1B_2$, and (4) $AB \approx BA$. Let \bar{A} denote the equivalence class of A and define $\bar{A} \cdot \bar{B} = \overline{AB}$. Then, by (3) and (4), the set \bar{S} of equivalence classes forms a commutative semi-group. Therefore \bar{S} has an invariant mean.

Let $m(S)$ be the space of bounded real-valued functions on S . If $x \in m(S)$ and $A \in S$ let x_A be the function in $m(S')$ defined by $x_A(A') = x(A'A)$ ($A' \in S'$). Let ν be a right invariant mean of S' and χ an invariant mean of \bar{S} . Then, for a fixed $x \in m(S)$, the function $\nu(x_A)$ ($A \in S$) is constant on \bar{A} . Therefore \bar{x} defined by $\bar{x}(\bar{A}) = \nu(x_A)$ is in $m(\bar{S})$. Then $\mu(x) = \chi(\bar{x})$ is a right invariant mean of S .

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QUASI-NIL RINGS

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Rings have been studied which have among others the property that every commutator $xy - yx$ is in the nucleus. It seems appropriate to consider rings in which the square of every element is in the nucleus, a property that is shared by both associative and Lie rings. Under the additional assumptions of primeness and characteristic different from 2 it can be shown that such rings are either associative or have the property that $x^2 = 0$, for every element x of the ring. If further $(x, y, z) + (y, z, x) + (z, x, y)$ is in the nucleus for all elements x, y, z of the ring, then the ring is either associative or a Lie ring.

We use the notation $(x, y, z) = (xy)z - x(yz)$. The nucleus N of a ring R consists of all $n \in R$ such that $(n, R, R) = (R, n, R) = (R, R, n) = 0$. N is a subring of R .

LEMMA. *Let R be a prime ring satisfying $x^2 \in N$ for every $x \in R$ and of characteristic different from 2. Then either R is associative or $N^2 = 0$.*

PROOF. For all $r, s \in R$, $rs + sr = (r + s)^2 - r^2 - s^2$ must be in N . Select $n, n' \in N$, and $x, y, z \in R$. Then $(n(n'x + xn'), y, z) = 0$, so that $(nn'x, y, z) = -(nxn', y, z)$. Similarly $(nxn', y, z) = -(xnn', y, z)$ and $(xnn', y, z) = -(nn'x, y, z)$. By combining these three equalities it follows that $2(nn'x, y, z) = 0$. Assuming characteristic not 2 it then follows that $(nn'x, y, z) = 0$. Since $(nx, y, z) = ((nx)y)z - (nx)(yz) = (n(xy)z - n(x(yz))) = n((xy)z) - n(x(yz)) = n(x, y, z)$, we replace n

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