APPLICATIONS OF DUALITY

EBEN MATLIS

Let R and S be two rings with units; then Cartan and Eilenberg have established the following duality identities (1) and (2) [**3**, Chapter 5, §5]. In the situation described by the symbol $({}_RA, {}_SB_R, {}_SC)$, where C is S-injective, we have an identity:

$$\text{(1)} \qquad \operatorname{Ext}_R(A, \operatorname{Hom}_S(B, C)) \cong \operatorname{Hom}_S(\operatorname{Tor}^R(B, A), C).$$

In the situation described by the symbol $({}_RA, {}_RB_S, C_S)$, if R is left Noetherian, A is finitely R-generated, and C is S-injective, then we have a second identity:

$$\text{(2)} \qquad \operatorname{Hom}_S(\operatorname{Ext}_R(A, B), C) \cong \operatorname{Tor}^R(\operatorname{Hom}_S(B, C), A).$$

In this note we shall apply these identities to the solution of two entirely dissimilar problems. First we shall prove that if R is a left Noetherian ring, then f.l.inj. dim $R=$ f.r.w. dim R. Secondly, we shall give a new, short proof of a theorem of Kaplansky [**5**] which states that a ring R is a Prufer ring if and only if, the torsion submodule of any finitely generated R-module is a direct summand.

Definition. Let R be a ring with unit. If A is a left R-module, we shall denote the injective dimension of A as an R-module by inj. $\dim_R A$. If B is a right R-module, we shall denote the weak dimension of B as an R-module by w. $\dim_R B$. We then define the finitistic, left, injective, dimension of R by f.l.inj. dim $R=$ sup. inj. $\dim_R A$, where A ranges over all left R-modules of finite injective dimension. Similarly, we define the finitistic, right, weak dimension of R by f.r.w. dim $R=$ sup.w. $\dim_R B$, where B ranges over all right R-modules of finite weak dimension.

Lemma 1. *Let A be a left module over a ring R. Then* inj. $\dim_R A \leqq n$ *if and only if*, $\operatorname{Ext}_R^{n+1}(R/I, A)=0$ *for every left ideal I of R.*

Proof. If inj. $\dim_R A \leqq n$, then of course $\operatorname{Ext}_R^{n+1}(X, A)=0$ for all left R-modules X. Conversely, suppose that $\operatorname{Ext}_R^{n+1}(R/I, A)=0$ for all left ideals I of R. Let

$$0 \to A \to C_0 \to C_1 \to \cdots \to C_{n-1} \to D \to 0$$

be an exact sequence of left R-modules, where C_i is R-injective for $i=0, \cdots, n-1$. Then $0=\operatorname{Ext}_R^{n+1}(R/I, A) \cong \operatorname{Ext}_R^1(R/I, D)$ for all left

Received by the editors November 12, 1958.

ideals I of R. Hence D is R-injective [3, Proposition 1.3.2], and thus inj. $\dim_R A \leqq n$.

Lemma 1 provides an easy proof of the following theorem of M. Auslander [1, Theorem 1]. (It has been brought to my attention that this proof was known, but unpublished, by S. Eilenberg).

Corollary. *For any ring R, l.gl.* dim $R = \sup. hd_R R/I$, *where I ranges over the left ideals of R.*

Proof. If $\sup. hd_R R/I = \infty$, then l.gl. dim $R = \infty$. Hence assume that $\sup. hd_R R/I = n$. Then $\operatorname{Ext}_R^{n+1}(R/I, A) = 0$ for all left R-modules A and all left ideals I of R. Thus by Lemma 1 inj. $\dim_R A \leqq n$ for all left R-modules A, and thus l.gl. dim $R \leqq n$.

The following lemma is well known, but we will prove it for the sake of completeness.

Lemma 2. *Let B be a right module over a ring R. Then* w. $\dim_R B \leqq n$ *if and only if*, $\operatorname{Tor}_{n+1}(B, R/I) = 0$ *for every left ideal I of R.*

Proof. If w. $\dim_R B \leqq n$, then by definition $\operatorname{Tor}_{n+1}^R(B, X) = 0$ for every left R-module X. Conversely, assume that $\operatorname{Tor}_{n+1}^R (B, R/I) = 0$ for every left ideal I of R. We assume by induction that $\operatorname{Tor}_{n+1}^R (B, Y) = 0$ for every left R-module Y generated by fewer than m elements. Let E be a left R-module generated by m elements, and E_1 a submodule of E generated by one of these elements. Then E/E_1 is generated by fewer than m elements. Hence from the exact sequence:

$$0 \to E_1 \to E \to E/E_1 \to 0$$

we obtain the exact sequence:

$$0 = \operatorname{Tor}_{n+1}^R(B, E_1) \to \operatorname{Tor}_{n+1}^R(B, E) \to \operatorname{Tor}_{n+1}^R(B, E/E_1) = 0.$$

Thus $\operatorname{Tor}_{n+1}^R (B, E) = 0$ for all finitely generated left R-modules E. Since Tor commutes with direct limits, $\operatorname{Tor}_{n+1}^R (B, X) = 0$ for all left R-modules X. Thus w. $\dim_R B \leqq n$.

Definition. Let Z denote the integers, and K the rationals (mod one). If X is a left (or right) module over a ring R, then $X^* = \operatorname{Hom}_Z(X, K)$ is a right (or left) module over R. It is easily seen that $X = 0$, if and only if $X^* = 0$.

Lemma 3. *Let R be any ring. Then* f.r.w. dim $R \leqq$ f.l.inj. dim R.

Proof. Let B be any right R-module, and A any left R-module. Then by (1) $\operatorname{Ext}_R(A, B^*) \cong \operatorname{Tor}^R(B, A)^*$. Thus w. $\dim_R B$

$= \text{inj.}\dim_R B^*$. It follows immediately that $\text{f.r.w.}\dim R \leqq \text{f.l.inj.}\dim R$.

THEOREM 1. *Let R be a left Noetherian ring. Then* $\text{f.l.inj.}\dim R = \text{f.r.w.}\dim R$.

PROOF. Let I be a left ideal of R, and A a left R-module. Then by (2) $\text{Ext}_R(R/I, A)^* \cong \text{Tor}^R(A^*, R/I)$. Hence by Lemmas 1 and 2, $\text{inj.}\dim_R A = \text{w.}\dim_R A^*$. Thus $\text{f.l.inj.}\dim R \leqq \text{f.r.w.}\dim R$. The opposite inequality is provided by Lemma 3.

Of course if R is a commutative, Noetherian ring, then there is no distinction between left and right modules; and then $\text{f.inj.}\dim R = \text{f.w.}\dim R$. It then follows from [**2**, Theorem 2.4] that $\text{f.inj.}\dim R \leqq \text{Krull}\dim R$.

As a final application of duality we give a new proof of a theorem of Kaplansky [**5**].

DEFINITION. A *Prufer ring* is an integral domain in which every finitely generated ideal is invertible.

THEOREM 2. *Let R be an integral domain. Then R is a Prufer ring if and only if the torsion submodule of every finitely generated R-module is a direct summand.*

PROOF. If R is a Prufer ring, a finitely generated, torsion-free R-module is projective, and the condition follows. Conversely, assume that the torsion submodule of every finitely generated R-module is a direct summand. Let S be a finitely generated, torsion-free R-module, and I an ideal of R. By [**4**, Theorem 2] and Lemma 2 it is sufficient to prove that $\text{Tor}_1^R(S, R/I) = 0$.

Suppose that $\text{Tor}_1^R(S, R/I) \neq 0$. Since $\text{Tor}_1^R(S, R/I)$ is a torsion module, there exists a torsion, injective R-module C such that $\text{Hom}_R(\text{Tor}_1^R(S, R/I), C) \neq 0$. By (1) we have $\text{Ext}_R^1(S, \text{Hom}_R(R/I, C)) \cong \text{Hom}_R(\text{Tor}_1^R(S, R/I), C)$. Since $\text{Hom}_R(R/I, C)$ is a torsion module, to establish a contradiction it will be sufficient to prove that $\text{Ext}_R^1(S, R) = 0$ for any torsion module T.

Let A be an extension of T by S. Now S is generated by a finite number of elements $\{y_1, \cdots, y_n\}$. Choose $x_i \in A$ such that $x_i \to y_i$, and let $B = (x_1, \cdots, x_n) \subset A$. Then $T \cap B$ is the torsion submodule of B. Hence by assumption $B = (T \cap B) \oplus S'$, where $S' \cong S$. Then $A = T \oplus S'$, and thus $\text{Ext}_R^1(S, T) = 0$.

BIBLIOGRAPHY

1. M. Auslander, *On the dimension of modules and algebras* (III), Nagoya Math. J. vol. 9 (1955) pp. 67–77.

2. M. Auslander and D. Buchsbaum, *Homological dimension in Noetherian rings*, II, Trans. Amer. Math. Soc. vol. 88 (1958) pp. 194–206.

3. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, 1956.

4. A. Hattori, *On Prufer rings*, J. Math. Soc. Japan vol. 9 no. 4 (1957) pp. 381–385.

5. I. Kaplansky, *A characterization of Prufer rings*, J. Indian Math. Soc. (forthcoming).

Northwestern University