ON MONOTONE AND POSITIVE SOLUTIONS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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1. **Introduction.** Second order nonlinear differential equations of the type

$$y'' + f(y, y')y' + g(y) = 0 \qquad (y' = dy/dx) \tag{1.1}$$

were first investigated by Levinson and Smith [**4**], and more recently by Utz [**5**; **6**; **7**; **8**], Antosiewicz [**1**], G. Brauer [**2**] and others. For a detailed treatment see Lefschetz [**3**]. The purpose of this note is to consider the behavior of solutions of (1.1) for large values of the independent variable, assuming the existence of such solutions. A solution is said to be oscillatory if it has positive maxima and negative minima for arbitrarily large values of x.

2. **Montone solutions.** The following result gives sufficient conditions for monotone solutions of (1.1)

THEOREM 1. *Let the following conditions be satisfied:*
(i) $\phi(y, y') = f(y, y')y'$, $f(y, y') \geqq 0$ *for all* y; y',
(ii) $yg(y) \geqq 0$ *for* $y' \neq 0$,
(iii) $\int_0^y g(z)dz \to \infty$ *as* $y \to \infty$,
(iv) $\int_E \{[\phi(y, y') + g(y)]/y'\} dx \neq \infty$,
where $E = \{x \mid y(x) \geqq 0,\ y'(x) \geqq 0\}$, *then a solution* $y(x)$, *valid for all large* x, *approaches zero monotonically as* x *tends to infinity.*

H. A. Antosiewicz [**1**] has shown that (i), (ii) and (iii) imply that $|y(x)|$ and $|y'(x)|$ remain bounded as $x \to \infty$. In [**5**] W. R. Utz established that (i), (ii), and (iii) also imply that if $y(x)$ is a solution of (1.1) which does not vanish identically, then $y(x)$ is bounded and oscillatory as $x \to \infty$, or $y(x)$ approaches zero monotonically as $x \to \infty$. Let us assume that a solution $y(x)$ of (1.1) does not approach zero monotonically as $x \to \infty$; then $y(x)$ must be oscillatory.

Let x_1 be a zero of $y(x)$ such that y is negative immediately to the left and positive immediately to the right of x_1. Then $y'(x_1) \geqq 0$. Moreover, if $y'(x_1) = 0$, then $y'(x) > 0$ on the immediate right of x_1 and consequently, $y''(x_1) > 0$ on the immediate right of x_1. But this is impossible as (1.1) shows. Hence $y'(x_1) > 0$. Let $\hat{x}_1$ denote the first zero of $y'(x)$ to the right of x_1. By integrating (1.1) over the interval $(x_1, \hat{x}_1)$ we have

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$$y'(\hat{x}_1) - y'(x_1) + \int_{x_1}^{\hat{x}_1} [\phi(y, y') + g(y)]dx = 0. \tag{2.1}$$

Since $y'(\hat{x}_1) = 0$,

$$y'(x_1) = \int_{x_1}^{\hat{x}_1} [\phi(y, y') + g(y)]dx. \tag{2.2}$$

By (ii) $g(y) > 0$ for $x_1 < x < \hat{x}_1$. Also $\phi(y, y') \geqq 0$, and hence $y''(x) < 0$. Hence y' is positive and decreasing on $(x_1, \hat{x}_1)$. Thus, by (2.2) we have

$$\begin{aligned} 1 &= \int_{x_1}^{\hat{x}_1} [\phi(y, y') + g(y)]dx/y'(x_1) \\ &\leqq \int_{x_1}^{\hat{x}_1} \{[\phi(y, y') + g(y)]/y'(x)\}dx. \end{aligned} \tag{2.3}$$

It follows from (iv) that the right-hand side of (2.3) tends to zero as x_1 tends to infinity. Hence, we arrive at a contradiction and $y(x)$ is monotone decreasing to zero as $x \to \infty$.

3. **On positive solutions.** The following theorem gives sufficient conditions for the nonexistence of solutions which remain positive for large values of x.

THEOREM 2. *If* (i) *and* (ii) *of Theorem* 1 *hold, and*
(v) $|\phi(y, y')| \leqq g(y)$ *for all* y, y',
(vi) *there exist numbers* a *and* b *such that* $a < b$ *and*

(vii)
$$\int_b^\infty (x - a)[\phi(y, y') + g(y)]dx = \infty,$$
$$(\phi(y, y') + g(y))^{1/2} \geqq [y(x)]^2$$

then no solution of (1.1) *can remain positive for all* x *greater than* a.

Suppose that there exists a solution $y(x)$ such that $y(x) > 0$ for $x > a$. It follows from (1.1) that $y''(x) < 0$, so $y'(x)$ is decreasing for $x > a$, and $y'(x)$ tends either to a finite limit or to $-\infty$. Moreover this limit cannot be negative for then $y(x)$ would become negative. Hence $y(x)$ must be ultimately nondecreasing, and $y'(x)$ tends to a finite non-negative limit. If we take $x > a$ and integrate (1.1) over the interval (x, ∞), we obtain:

$$y'(\infty) - y'(x) + \int_x^\infty [\phi(y, y') + g(y)]dx = 0. \tag{3.1}$$

(The convergence of the integral is guaranteed by the existence of

$y'(\infty)=\lim_{x\to\infty} y(x)$.) Since $y'(\infty)\geqq 0$, we have, by (3.1)

$$y'(x) \geqq \int_x^\infty [\phi(y, y') + g(y)]dx. \tag{3.2}$$

Integrating (3.2) over the interval (a, x) for $x>a$ we have

$$y(x) - y(a) \geqq \int_a^x \int_u^\infty [\phi(y, y') + g(y)]dtdu \tag{3.3}$$

and thus

$$y(x) \geqq \int_a^x (t-a)[\phi(y, y') + g(y)]dt. \tag{3.4}$$

From (3.4) and by making use of (vii), we have the following inequalities,

$$\begin{aligned} &\frac{(x-a)[\phi(y, y') + g(y)]^{1/2}}{\left\{\int_a^x (t-a)[\phi(y,y') + g(y)]dt\right\}^2} \\ &\qquad \geqq \frac{[y(x)]^2(x-a)}{\left\{\int_a^x (t-a)[\phi(y,y') + g(y)]dt\right\}^2} \geqq x-a. \end{aligned} \tag{3.5}$$

If we multiply both sides of (3.5) by $[\phi(y,y')+g(y)]^{1/2}$, choose x_1 such that $a<x_1$, and integrate (3.5) from b to x_1 we have

$$\begin{aligned} &\int_b^{x_1} \left\{\int_a^x (t-a)[\phi(y,y')+g(y)]dt\right\}^{-2} \{(x-a)[\phi(y,y')+g(y)]\} dx \\ &\qquad \geqq \int_b^{x_1} (t-a)[\phi(y, y') + g(y)]^{1/2}dt. \end{aligned} \tag{3.6}$$

As $x_1\to\infty$ the left-hand side of (3.6) remains bounded, hence we reach a contradiction to (vi). Thus $y(x)$ cannot remain positive for all large values of x.

REFEREE'S REMARK. The author's proof of Theorem 2 also shows that $y(x)$ cannot be negative for all large values of x. Thus $y(x)$ changes sign at arbitrarily large x-values, that is, y is oscillatory.

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