ON AUTOMORPHIC-INVERSE PROPERTIES IN LOOPS

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Introduction. In a loop $(G, \cdot)$ we define J as the mapping that takes every element into its right inverse, i.e., $x \cdot xJ = 1$, for all x in G. It is well known [**3**] that in I.P. loops J is an anti-automorphism. In crossed-inverse loops (in short *C.I. loops*), which are defined as satisfying either of the equivalent identities $xy \cdot xJ = y$ or $x(y \cdot xJ) = y$, for all x and y in G, J is known [**1**] to be an automorphism. On the other hand, easily constructed counter-examples show that a loop in which J is an anti-automorphism is not necessarily I.P., and that a loop in which J is an automorphism is not necessarily C.I. Furthermore a loop which is I.P. *everywhere*, i.e. all of whose isotopes are I.P., is known [**4**] to be Moufang. A loop which is C.I. everywhere is an abelian group; this can be easily checked by computation, but it is also obvious, due to the fact that the C.I. property can be represented in a 3-web by the validity of the Thomsen figure (cf. [**7**]) in a special position while the general Thomsen figure corresponds to both associativity and commutativity.

It is the purpose of this paper to show that an I.P. loop in which J is an anti-automorphism everywhere is Moufang, and that a C.I. loop in which J is an automorphism everywhere is an abelian group. Thus we show that the weaker "automorphic-inverse properties" $(xy)J = yJ \cdot xJ$ and $(xy)J = xJ \cdot yJ$ in I.P. loops and C.I. loops, respectively, are sufficient to replace the full I.P. and C.I. properties of the isotopes.

THEOREM 1. *An I.P. loop, in all of whose isotopes J is an anti-automorphism, is Moufang.*

PROOF. We consider an isotope with $xg * y = xy$. Its unit is g. Let xJ^* be the right inverse of x in the isotope. Using the inverse property, which implies $J = J^{-1}$, we have then

$$xg^{-1} \cdot xJ^* = g, \quad \text{and} \quad J^* = JL(g)R(g).$$

The theorem assumes $(x * y)J^* = yJ^* * xJ^*$, that is $g(xg^{-1} \cdot y)^{-1} \cdot g = [(gy^{-1} \cdot g)g^{-1}](gx^{-1} \cdot g)$. With $gx^{-1} = a$, $y^{-1} = b$ we get $(g \cdot ba)g = gb \cdot ag$, for all a, b, g in the loop. Thus the loop is Moufang.

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THEOREM 2. *A C.I. loop, G, in all of whose isotopes J is an automorphism, is an abelian group.*

PROOF. The proof will be performed in 5 stages, (i) to (v).

(i) *The inverses in G are unique.*

Let the isotope be defined by $xg * fy = xy$. Then we have, using the C.I. property,

$$(gJ^{-1}\cdot x)(xJ^*\cdot fJ) = fg, \qquad \text{and} \qquad J^* = JL(g)L(fg)L(f).$$

The identity $(x * y)J^* = xJ^* * yJ^*$ becomes

$$(1)\quad f[fg\cdot(g((g\cdot xJ)(yJ\cdot fJ^2)))] = [gJ^{-1}\cdot(f(fg\cdot(g\cdot xJ)))][fg\cdot(g\cdot yJ)].$$

Putting $x = g^2g$ and $y = f = 1$, yields $1 = (gJ^{-1}\cdot gJ)g^2$, and finally $gJ^{-1} = gJ$. Thus inverses are unique in G.

(ii) *The squares form a normal subloop and lie in the centre.*

If we substitute in (1) $gx^{-1} = a$, $y^{-1}f = b$ we get

$$f[fg\cdot(g\cdot ab)] = [g^{-1}(f(fg\cdot a))][fg\cdot(g\cdot f^{-1}b)].$$

This is an autotopism (cf. [4])

$$(2)\qquad (L(fg)L(f)L(g^{-1}),\ L(f^{-1})L(g)L(fg),\ L(g)L(fg)L(f)).$$

With $f = 1$ this becomes

$$(3)\qquad (L(g)L(g^{-1}),\ L(g)L(g),\ L(g)L(g)).$$

According to [2] every autotopism (U, V, W) in a C.I. loop implies another autotopism $(L(1U), L(1V), L(1W))$. Thus (3) implies $(I, L(g^2), L(g^2))$ as a new autotopism, that is,

$$(4)\qquad a\cdot g^2b = g^2\cdot ab.$$

With $b = 1$ this yields $ag^2 = g^2a$, hence the squares of the elements commute with all elements of G. Equation (4) becomes $a\cdot bg^2 = ab\cdot g^2$. Thus all the squares belong to the right nucleus, but, as the author has proved [2], all the elements of the right nucleus are centre elements, and therefore all squares of loop elements lie in the centre. Moreover the squares form a normal subloop:

$$x^2y^2 = x^2y^2(xy\cdot x^{-1}y^{-1}) = xy\cdot(x^2x^{-1}\cdot y^2y^{-1}) = xy\cdot xy = (xy)^2,$$

and $(x^2)J = (xJ)^2$.

(iii) *G is commutative.*

According to [2], the elements $1U$ and $1V$, for all autotopisms (U, V, W) of G, form a Moufang subloop, and the cubes of the elements of this subloop lie in the centre of G. Since also the squares lie in the centre, $1U$ and $1V$ themselves lie in the centre. In particular take as (U, V, W) the autotopism (2), then $1U=g^{-1}(f\cdot fg)$ and $1V=fg\cdot gf^{-1}$ lie in the centre of G. Therefore we have

$$g^{-1}(g^{-1}f\cdot 1V) = (g^{-1}\cdot g^{-1}f)\cdot 1V,$$

$$f = g^{-1}(g^{-1}f\cdot(fg\cdot gf^{-1})) = (g^{-1}\cdot g^{-1}f)(fg\cdot gf^{-1}),$$

$$f(g\cdot gf^{-1}) = fg\cdot gf^{-1} \tag{5}$$

and

$$1V\cdot f^{-1}g^{-1} = f^{-1}g^{-1}\cdot 1V,$$

$$gf^{-1} = (fg\cdot gf^{-1})\cdot f^{-1}g^{-1} = f^{-1}g^{-1}\cdot(fg\cdot gf^{-1}),$$

$$gf^{-1}\cdot fg = fg\cdot gf^{-1}. \tag{6}$$

We interchange f and g in (5) and get

$$g(f\cdot fg^{-1}) = gf\cdot fg^{-1} = (gf^{-1}\cdot fg)f^2g^{-2}.$$

In view of (6) this becomes

$$g^{-1}(f\cdot fg) = (fg\cdot gf^{-1})f^2g^{-2}, \text{ or } 1U = 1V\cdot f^2g^{-2}.$$

Now obviously $1W=1U\cdot 1V$, hence $1W=1V\cdot f^2g^{-2}\cdot 1V$, that is, $f(fg\cdot g)=f^2g^{-2}(fg\cdot gf^{-1})^2$, and using the multiplication rule for squares, $f(fg\cdot g)=f^2g^{-2}f^2g^2g^2f^{-2}=f^2g^2$, and $fg=f^2g^2\cdot g^{-1}f^{-1}=gf$. Thus the commutativity of G is established.

The commutativity and the C.I. property imply the inverse property. Moreover the alternative property holds:

$$a\cdot ab = a^2(a\cdot a^{-1}b) = a^2b.$$

(iv) *G is an A-loop.*

Using the associativity of multiplication by squares, we may write the autotopism (2)

$$(U, V, W) = (L(fg)^{-1}L(f)L(g),\ L(f)L(g)L(fg)^{-1},\ L(g)L(fg)^{-1}L(f)).$$

Now $1U=1V=1$, and that implies $U=V=W$. Hence $V=L(f)L(g)L(fg)^{-1}$ is an automorphism. We shall prove now that every inner mapping of G is the product of inner mappings of the form $L(f)L(g)L(fg)^{-1}$. Owing to the commutativity of G and the property $L^{-1}(x)=L(x^{-1})$ of I.P. loops, every inner mapping S of G has the form $S=\prod_{k=1}^{n} L(a_k)$, with $1S=1$. We can write

$$S = [L(a_1)L(a_2)L(a_1a_2)^{-1}][L(a_1a_2)L(a_3)L(a_1a_2\cdot a_3)^{-1}] \cdots \left[L\left(1\prod_{k=1}^{n-1}L(a_k)\right)L(a_n)L\left(1\prod_{k=1}^{n}L(a_k)\right)^{-1}\right]$$

because the last term of each bracket and the first term of the next bracket are of the type $L(x)L(x^{-1})$ and cancel out; furthermore the last term of the last bracket is

$$L\left(1\prod_{k=1}^{n} L(a_k)\right)^{-1} = L(1S)^{-1} = L(1^{-1}) = I.$$

Each of the brackets contains a product of the form

$$V = L(f)L(g)L(fg)^{-1}$$

which is an automorphism. Hence every inner mapping S is a product of automorphisms, and therefore an automorphism. A loop in which every inner mapping is an automorphism is usually called an *A-loop*.

(v) *G is abelian.*

G is a commutative I.P. loop. A-loops with the inverse property are diassociative [5], and if they are also commutative they are Moufang [6]. Thus we have the identity $(xy\cdot x)z=x(y\cdot xz)$, and therefore $x^2y\cdot z=x(y\cdot xz)$, $yz=x^{-1}(y\cdot xz)$, and $yz\cdot x=y\cdot zx$. Hence G is associative and therefore abelian.

Remark. In recent work (to appear in Pacific J. Math.) J. M. Osborn has been dealing with a generalization of both C.I. and I.P. loops: "weak-inverse loops" in which $xy\cdot z=1$ if, and only if, $x\cdot yz=1$. In weak-inverse loops J^2 is an automorphism, but this property is not sufficient for defining weak-inverse loops. Osborn has investigated loops which are weak-inverse everywhere, and these loops have interesting properties. In connection with the present paper it might be of interest whether a weak-inverse loop all of whose isotopes have J^2 as an automorphism (and are not necessarily weak-inverse) has already the same properties, such that again an "automorphic-inverse property" would suffice to replace the full loop identity in the isotopes.

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