ON AUTOMORPHIC-INVERSE PROPERTIES IN LOOPS

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Introduction. In a loop \((G, \cdot)\) we define \(J\) as the mapping that takes every element into its right inverse, i.e., \(x \cdot xJ = 1\), for all \(x\) in \(G\). It is well known [3] that in I.P. loops \(J\) is an anti-automorphism. In crossed-inverse loops (in short C.I. loops), which are defined as satisfying either of the equivalent identities \(xy \cdot xJ = y\) or \(x(y \cdot xJ) = y\), for all \(x\) and \(y\) in \(G\), \(J\) is known [1] to be an automorphism. On the other hand, easily constructed counter-examples show that a loop in which \(J\) is an anti-automorphism is not necessarily I.P., and that a loop in which \(J\) is an automorphism is not necessarily C.I. Furthermore a loop which is I.P. everywhere, i.e. all of whose isotopes are I.P., is known [4] to be Moufang. A loop which is C.I. everywhere is an abelian group; this can be easily checked by computation, but it is also obvious, due to the fact that the C.I. property can be represented in a 3-web by the validity of the Thomsen figure (cf. [7]) in a special position while the general Thomsen figure corresponds to both associativity and commutativity.

It is the purpose of this paper to show that an I.P. loop in which \(J\) is an anti-automorphism everywhere is Moufang, and that a C.I. loop in which \(J\) is an automorphism everywhere is an abelian group. Thus we show that the weaker “automorphic-inverse properties” \((xy)J = yJ \cdot xJ\) and \((xy)J = xJ \cdot yJ\) in I.P. loops and C.I. loops, respectively, are sufficient to replace the full I.P. and C.I. properties of the isotopes.

Theorem 1. An I.P. loop, in all of whose isotopes \(J\) is an anti-automorphism, is Moufang.

Proof. We consider an isotope with \(xg \cdot y = xy\). Its unit is \(g\). Let \(xJ^*\) be the right inverse of \(x\) in the isotope. Using the inverse property, which implies \(J = J^{-1}\), we have then

\[
{xg^{-1} \cdot xJ^* = g, \quad \text{and} \quad J^* = JL(g)R(g).}
\]

The theorem assumes \((x \ast y)J^* = yJ^* \ast xJ^*\), that is \(g(xg^{-1} \cdot y)^{-1} \cdot g = [(gy^{-1} \cdot g)^{-1}g^{-1}](gx^{-1} \cdot g)\). With \(gx^{-1} = a\), \(y^{-1} = b\) we get \((g \cdot ba)g = gb \cdot ag\), for all \(a, b, g\) in the loop. Thus the loop is Moufang.

Presented to the Society, August 29, 1958; received by the editors July 6, 1958 and, in revised form, November 24, 1958.
Theorem 2. A C.I. loop, G, in all of whose isotopes \( J \) is an automorphism, is an abelian group.

Proof. The proof will be performed in 5 stages, (i) to (v).

(i) The inverses in \( G \) are unique.

Let the isotope be defined by \( xy * J = xy \). Then we have, using the C.I. property,

\[
(g^{-1} x)(xJ * yJ) = fg, \quad \text{and} \quad J^* = JL(g)L(fg)L(f).
\]

The identity \( (x * y)J^* = xJ^* * yJ^* \) becomes

\[
(1) \quad f[fg \cdot (g * (xJ)(yJ * J^2))] = [gJ^{-1} \cdot (f(fg \cdot (g * xJ)))] [fg \cdot (g \cdot yJ)]
\]

Putting \( x = g^2g \) and \( y = f = 1 \), yields \( 1 = (gJ^{-1} \cdot gJ)g^2 \), and finally \( gJ^{-1} = gJ \). Thus inverses are unique in \( G \).

(ii) The squares form a normal subloop and lie in the centre.

If we substitute in (1) \( gx^{-1} = a, y^{-1}f = b \) we get

\[
f[fg \cdot (g \cdot ab)] = [g^{-1}(f(fg \cdot a))] [fg \cdot (g \cdot f^{-1}b)].
\]

This is an autotopism (cf. [4])

\[
(2) \quad (L(fg)L(f)L(g^{-1}), L(f^{-1})L(g)L(fg), L(g)L(fg)L(f)).
\]

With \( f = 1 \) this becomes

\[
(3) \quad (L(g)L(g^{-1}), L(g)L(g), L(g)L(g)).
\]

According to [2] every autotopism \( (U, V, W) \) in a C.I. loop implies another autotopism \( (L(1U), L(1V), L(1W)) \). Thus (3) implies \( (I, L(g^2), L(g^2)) \) as a new autotopism, that is,

\[
(4) \quad a \cdot g^2b = g^2 \cdot ab.
\]

With \( b = 1 \) this yields \( ag^2 = g^2a \), hence the squares of the elements commute with all elements of \( G \). Equation (4) becomes \( a \cdot bg^2 = ab \cdot g^2 \).

Thus all the squares belong to the right nucleus, but, as the author has proved [2], all the elements of the right nucleus are centre elements, and therefore all squares of loop elements lie in the centre. Moreover the squares form a normal subloop:

\[
x^2y^2 = x^2y^2(xy \cdot x^{-1}y^{-1}) = xy \cdot x^2x^{-1}y^2y^{-1} = xy \cdot xy = (xy)^2,
\]

and \( (x^2)J = (xJ)^2 \).

(iii) \( G \) is commutative.

According to [2], the elements \( 1U \) and \( 1V \), for all autotopisms \( (U, V, W) \) of \( G \), form a Moufang subloop, and the cubes of the elements of this subloop lie in the centre of \( G \). Since also the squares lie in the centre, \( 1U \) and \( 1V \) themselves lie in the centre. In particular
take as \((U, V, W)\) the autotopism (2), then \(1U = g^{-1}(f \cdot fg)\) and \(1V = fg \cdot gf^{-1}\) lie in the centre of \(G\). Therefore we have

\[
g^{-1}(g^{-1}f \cdot 1V) = (g^{-1} \cdot g^{-1}f) \cdot 1V,
\]

\[
f = g^{-1}(g^{-1}f \cdot (fg \cdot gf^{-1})) = (g^{-1} \cdot g^{-1}f)(fg \cdot gf^{-1}),
\]

(5)

\[
f(g \cdot gf^{-1}) = fg \cdot gf^{-1}
\]

and

\[
1V \cdot f^{-1}g^{-1} = f^{-1}g^{-1} \cdot 1V,
\]

\[
gf^{-1} = (fg \cdot gf^{-1}) \cdot f^{-1}g^{-1} = f^{-1}g^{-1} \cdot (fg \cdot gf^{-1}),
\]

(6)

\[
gf^{-1} \cdot fg = fg \cdot gf^{-1}.
\]

We interchange \(f\) and \(g\) in (5) and get

\[
g(f \cdot gf^{-1}) = gf \cdot fg^{-1} = (gf^{-1} \cdot fg)g^{2}g^{-2}.
\]

In view of (6) this becomes

\[
g^{-1}(f \cdot fg) = (fg \cdot gf^{-1})g^{2}g^{-2}, \text{ or } 1U = 1V \cdot f^{2}g^{-2}.
\]

Now obviously \(1W = 1U \cdot 1V\), hence \(1W = 1V \cdot f^{2}g^{-2} \cdot 1V\), that is, \(f(fg \cdot g) = f^{2}g^{-2}(fg \cdot gf^{-1})^{2}\), and using the multiplication rule for squares, \(f(fg \cdot g) = f^{2}g^{-2}f^{2}g^{2}g^{2}f^{-2} = f^{2}g^{2}\), and \(fg = f^{2}g \cdot g^{-1}f^{-1} = gf\). Thus the commutativity of \(G\) is established.

The commutativity and the C.I. property imply the inverse property. Moreover the alternative property holds:

\[
a \cdot ab = a^{2}(a \cdot a^{-1}b) = a^{2}b.
\]

(iv) \(G\) is an \(A\)-loop.

Using the associativity of multiplication by squares, we may write the autotopism (2)

\[
(U, V, W) = (L(fg)^{-1}L(f)L(g), L(f)L(g)L(fg)^{-1}, L(g)L(fg)^{-1}L(f)).
\]

Now \(1U = 1V = 1\), and that implies \(U = V = W\). Hence \(V = L(f)L(g)L(fg)^{-1}\) is an automorphism. We shall prove now that every inner mapping of \(G\) is the product of inner mappings of the form 

\[
L(f)L(g)L(fg)^{-1}.
\]

Owing to the commutativity of \(G\) and the property 
\(L^{-1}(x) = L(x^{-1})\) of I.P. loops, every inner mapping \(S\) of \(G\) has the form 

\[
S = \prod_{k=1}^{n} L(a_{k}), \text{ with } 1S = 1.
\]

We can write 

\[
S = [L(a_{1})L(a_{2})L(a_{1}a_{2})^{-1}][L(a_{1}a_{2})L(a_{3})L(a_{1}a_{2} \cdot a_{3})^{-1}]
\]

\[
\cdots \left[L \left( \prod_{k=1}^{n-1} L(a_{k}) \right) L(a_{n})L \left( \prod_{k=1}^{n} L(a_{k}) \right)^{-1} \right]
\]

because the last term of each bracket and the first term of the next
bracket are of the type $L(x)L(x^{-1})$ and cancel out; furthermore the last term of the last bracket is
\[ L \left( 1 \prod_{k=1}^{n} L(a_k) \right)^{-1} = L(1S)^{-1} = L(1^{-1}) = I. \]

Each of the brackets contains a product of the form
\[ V = L(f)L(g)L(fg)^{-1} \]
which is an automorphism. Hence every inner mapping $S$ is a product of automorphisms, and therefore an automorphism. A loop in which every inner mapping is an automorphism is usually called an $A$-loop.

(v) $G$ is abelian.

$G$ is a commutative I.P. loop. $A$-loops with the inverse property are diassociative [5], and if they are also commutative they are Moufang [6]. Thus we have the identity $(xy \cdot x)z = x(y \cdot xz)$, and therefore $x^2y \cdot z = x(y \cdot xz)$, $yz = x^{-1}(y \cdot xz)$, and $yz \cdot x = y \cdot xz$. Hence $G$ is associative and therefore abelian.

Remark. In recent work (to appear in Pacific J. Math.) J. M. Osborn has been dealing with a generalization of both C.I. and I.P. loops: “weak-inverse loops” in which $xy \cdot z = 1$ if, and only if, $x \cdot yz = 1$.

In weak-inverse loops $J^2$ is an automorphism, but this property is not sufficient for defining weak-inverse loops. Osborn has investigated loops which are weak-inverse everywhere, and these loops have interesting properties. In connection with the present paper it might be of interest whether a weak-inverse loop all of whose isotopes have $J^2$ as an automorphism (and are not necessarily weak-inverse) has already the same properties, such that again an “automorphic-inverse property” would suffice to replace the full loop identity in the isotopes.

References


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