A RADICAL ALGEBRA WITHOUT DERIVATIONS

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Wermer and Singer [1] have shown that in a semi-simple commutative Banach algebra there exist no nontrivial derivations, a derivation being a bounded linear operator D, taking the algebra into itself, with the additional property that

$$D(u\cdot v) = u\cdot(Dv) + (Du)\cdot v.$$

Wermer has conjectured the following converse: If a commutative Banach algebra has no nontrivial derivations then it is semi-simple.

A weaker statement is: If a commutative Banach algebra is all radical [i.e. $x^n \to 0$ for all x] then it has a nontrivial derivation.

In this note we show that even this weaker statement is false.

We choose a fixed sequence λ_n, $n=1, 2, \cdots$, of non-0 complex numbers, and consider the following algebraic system $\mathcal{S}$.

The elements are those formal power series

$$a(t) = \sum_{n=1}^{\infty} a_n t^n \text{ for which } \sum |a_n|\,|\lambda_n|^n < \infty.$$

Addition, multiplication, and multiplication by scalars is as usual.

Consider now the following properties

A: $|\lambda_n| \geqq |\lambda_{n+1}|$, $n=1, 2, \cdots$,

B: $\lambda_n \to 0$,

C: $n^\epsilon |\lambda_{n+1}|^{n+1}/|\lambda_n|^n \to \infty$ for any fixed $\epsilon > 0$.

We prove the following lemmas:

LEMMA 1. A $\Rightarrow \mathcal{S}$ *is a Banach algebra under the norm* $\|a\| = \sum |a_n|\,|\lambda_n|^n$.

LEMMA 2. A *and* B$\Rightarrow\mathcal{S}$ *is all radical.*

LEMMA 3. A *and* C$\Rightarrow\mathcal{S}$ *has no nontrivial derivations.*

PROOF 1. Since $\mathcal{S}$, under this norm is isomorphic and isometric to l^1 under the correspondence

$$\sum_{n=1}^{\infty} a_n t^n \leftrightarrow \{a_n(\lambda_n)^n\},$$

$\mathcal{S}$ is clearly a Banach space. Also

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$$\begin{aligned}
\|ab\| &= \left\| \sum_{n=2}^{\infty} \left(\sum_{m+k=n} a_m b_k \right) t^n \right\| \\
&= \sum_{n=2}^{\infty} \left| \sum_{m+k=n} a_m b_k \right| \cdot |\lambda_n|^n \\
&\leqq \sum_{m,k} |a_m| \, |b_k| \, |\lambda_{m+k}|^{m+k} = \sum_{m,k} = |a_m| \, |b_k| \, |\lambda_{m+k}|^m |\lambda_{m+k}|^k \\
&\leqq \sum_{m,k} |a_m| \, |b_k| \, |\lambda_m|^m \cdot |\lambda_k|^k \text{ (by A.)} \\
&= \|a\| \cdot \|b\|
\end{aligned}$$

and so is a Banach algebra.

PROOF 2. It is known that the radical R, is a closed subalgebra. But now $\|t^n\|^{1/n} = |\lambda_n|$ and by B this $\to 0$. $\therefore t \in R$, by the algebraic closure of R, $P(t) \in R$, P any polynomial, $\therefore$ by topological closure, all

$$a(t) \in R \cdot \left[\|a(t) - (a_1 + \cdots a_N t^N)\| = \sum_{n=N+1}^{\infty} |a_n| \, |\lambda_n|^n \to 0 \text{ with } N \right].$$

PROOF 3. Since, as in the above parenthetical remark, the polynomials are dense in $\mathcal{S}$ it suffices to prove that $D(t) = 0$ for D any derivation, for it would then follow that $D(P(t)) = P'(t)Dt = 0$ and so $D(a(t)) = 0$ or D is trivial.

Let

$$D(t) = \sum_{m=1}^{\infty} c_m t^m.$$

Then

$$D(t^n) = nt^{n-1} D(t) = n \sum_{k=0}^{\infty} c_{k+1} t^{k+n}.$$

$\therefore$ for any fixed $k = 0, 1, \cdots$

$$\|D(t^n)\| \geqq n \, |c_{k+1}| \, |\lambda_{k+n}|^{k+n},$$

on the other hand, D being bounded,

$$\begin{aligned}
\|D(t^n)\| &\leqq M \|t^n\| = M \, |\lambda_n|^n, \\
\therefore \; |c_{k+1}| &\leqq M \, |\lambda_n|^n / n \, |\lambda_{n+k}|^{n+k}.
\end{aligned}$$

Now as $n \to \infty$ it follows from C that the right side $\to 0$, $\therefore c_{k+1} = 0$,

this holding for each $k=0, 1, \cdots$ it follows that $D(t)=0$.

We now see that a counterexample to Wermer's conjecture is afforded us once we note that conditions A, B, C are not contradictory. This is clear, however, since e.g.

$$\lambda_n = \frac{1}{\log(n+1)}$$

satisfies all three of them.

Reference

1. I. M. Singer and J. Wermer, *Derivations on commutative normed algebras*, Math. Ann. vol. 129 (1955) pp. 260–264.

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