A RADICAL ALGEBRA WITHOUT DERIVATIONS
DONALD J. NEWMAN

Wermer and Singer [1] have shown that in a semi-simple commuta-
tive Banach algebra there exist no nontrivial derivations, a deriva-
tion being a bounded linear operator \( D \), taking the algebra into it-
self, with the additional property that

\[
D(u \cdot v) = u \cdot (Dv) + (Du) \cdot v.
\]

Wermer has conjectured the following converse: If a commutative
Banach algebra has no nontrivial derivations then it is semi-simple.
A weaker statement is: If a commutative Banach algebra is all
radical [i.e. \( x^n \rightarrow 0 \) for all \( x \)] then it has a nontrivial derivation.
In this note we show that even this weaker statement is false.
We choose a fixed sequence \( \lambda_n, n = 1, 2, \cdots \), of non-0 complex
numbers, and consider the following algebraic system \( S \).
The elements are those formal power series

\[
a(t) = \sum_{n=1}^{\infty} a_n t^n \text{ for which } \sum |a_n| |\lambda_n|^n < \infty.
\]

Addition, multiplication, and multiplication by scalars is as usual.
Consider now the following properties
A: \(|\lambda_n| \geq |\lambda_{n+1}|\), \( n = 1, 2, \cdots \),
B: \(\lambda_n \rightarrow 0\),
C: \(n^\epsilon |\lambda_{n+1}|^{n+1}/|\lambda_n|^n \rightarrow \infty\) for any fixed \( \epsilon > 0 \).
We prove the following lemmas:

**Lemma 1.** A \( \Rightarrow \) \( S \) is a Banach algebra under the norm \(||a|| = \sum |a_n| |\lambda_n|^n\).

**Lemma 2.** A and B \( \Rightarrow S \) is all radical.

**Lemma 3.** A and C \( \Rightarrow S \) has no nontrivial derivations.

**Proof 1.** Since \( S \), under this norm is isomorphic and isometric to
\( l^1 \) under the correspondence

\[
\sum_{n=1}^{\infty} a_n t^n \leftrightarrow \{ a_n(\lambda_n)^n \},
\]

\( S \) is clearly a Banach space. Also

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\[
\|ab\| = \left\| \sum_{n=2}^{\infty} \left( \sum_{m+k=n} a_m b_k \right) t^n \right\|
\]
\[
= \sum_{n=2}^{\infty} \left| \sum_{m+k=n} a_m b_k \right| \|\lambda_n\|^n
\]
\[
\leq \sum_{m,k} \|a_m\| \|b_k\| \|\lambda_{m+k}\|^m \|\lambda_{m+k}\|^k = \sum_{m,k} \|a_m\| \|b_k\| \|\lambda_m\|^m \|\lambda_k\|^k \quad \text{(by A.)}
\]
\[
= \|a\| \|b\|
\]
and so is a Banach algebra.

**Proof 2.** It is known that the radical \( R \) is a closed subalgebra. But now \( \|t^n\|^{1/n} = \|\lambda_n\| \) and by B this \( \to 0 \). \( \therefore t \in R \), by the algebraic closure of \( R \), \( P(t) \in R \), \( P \) any polynomial. \( \therefore \) by topological closure, all

\[
a(t) \in R \cdot \left[ \|a(t) - (a_{1t} + \cdots + a_{Nt^N})\| = \sum_{n=N+1}^{\infty} |a_n| \|\lambda_n\|^n \to 0 \text{ with } N \right].
\]

**Proof 3.** Since, as in the above parenthetical remark, the polynomials are dense in \( S \) it suffices to prove that \( D(t) = 0 \) for \( D \) any derivation, for it would then follow that \( D(P(t)) = P'(t)Dt = 0 \) and so \( D(a(t)) = 0 \) or \( D \) is trivial.

Let

\[
D(t) = \sum_{m=1}^{\infty} c_m t^m.
\]

Then

\[
D(t^n) = n t^{n-1} D(t) = n \sum_{k=0}^{\infty} c_{k+1} t^{k+n}.
\]

\( \therefore \) for any fixed \( k = 0, 1, \cdots \)

\[
\|D(t^n)\| \geq n |c_{k+1}| |\lambda_{k+n}|^{k+n},
\]
on the other hand, \( D \) being bounded,

\[
\|D(t^n)\| \leq \|M\| t^n = M |\lambda_n|^{n},
\]

\( \therefore |c_{k+1}| \leq M |\lambda_n|^{n/n} |\lambda_{n+k}|^{n+k}. \)

Now as \( n \to \infty \) it follows from C that the right side \( \to 0 \), \( \therefore c_{k+1} = 0 \),

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this holding for each $k = 0, 1, \cdots$ it follows that $D(t) = 0$.

We now see that a counterexample to Wermer's conjecture is afforded us once we note that conditions A, B, C are not contradictory. This is clear, however, since e.g.

$$\lambda_n = \frac{1}{\log(n + 1)}$$

satisfies all three of them.

**Reference**


Massachusetts Institute of Technology and Brown University