WEAK n-HOMOGENEITY IMPLIES WEAK $(n-1)$-HOMOGENEITY

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Introduction and definitions. Let $\tilde{S}$ be a set of points. By an *n-subset* of $\tilde{S}$ we shall mean a subset of $\tilde{S}$ consisting of precisely n points. Let S be a subset of $\tilde{S}$ and G a group of 1-1 transformations of $\tilde{S}$ upon itself. Then G is *weakly n-transitive* over S if for any two n-subsets X_1, X_2 of S there exists a $g \in G$ such that $g(X_1) = X_2$. (It is not required that $g(S) \subset S$.) In particular if $S = \tilde{S}$, $\tilde{S}$ is a topological space, and G is the group of homeomorphisms of $\tilde{S}$ upon itself, then the space S is said to be *weakly n-homogeneous*.[1]

The purpose of this paper is to prove that an infinite weakly n-homogeneous topological space is weakly $(n-1)$-homogeneous $(n > 1)$.

THEOREM 1. *Let $\tilde{S}$ be a set and let n, k be fixed positive integers $(1 < n \leqq k)$. Let S be a k-subset of $\tilde{S}$ and let G be weakly n-transitive over S. Then a sufficient condition that G be weakly $(n-1)$-transitive over S is that g.c.d. $(n, (C_{k,n-1})) = 1$.*

PROOF. Let $X_1, X_2, \cdots, X_{(C_{k,n})}$ be the $(C_{k,n})$ different n-subsets of S. We shall impose a convenient ordering on the points of each X_i. Let $X_1 = a_{11} \cup a_{12} \cup \cdots \cup a_{1n}$. Let V_1 be the ordered set

$$\langle a_{11}, a_{12}, \cdots, a_{1n} \rangle.$$

Now for each X_i there is a g_i in G such that $g_i(X_1) = X_i$. Let V_i be the ordered set $\langle g_i(a_{11}), g_i(a_{12}), \cdots, g_i(a_{1n}) \rangle$. It will be convenient to envision the $(C_{k,n}) \times n$ matrix

$$A = \begin{bmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \\ \cdot \\ V_{(C_{k,n})} \end{bmatrix}.$$

The transformation g_i may be thought of as a projection of the first

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[1] The definitions of higher homogeneity in the literature differ. The referee has suggested the following terminology (which the author adopts in this paper): n-homogeneity refers to *ordered* n-tuples, weak n-homogeneity refers to n-sets. This terminology coincides with the more standard concepts of transitivity and weak transitivity of groups operating on a set.

row of A onto the ith row. The transformation $g_j g_i^{-1}$ projects the ith row onto the jth row.

For $r=1, 2, \cdots, n$ let A_r be the $(C_{k,n})\times(n-1)$ matrix obtained by striking out the rth column of A. Let $\{M\}$ denote the class of different $(n-1)$-subsets of S. Suppose $M\in\{M\}$. For notational purposes we shall say that M *occurs as a row* in A_r if the elements of M are the elements of some row of A_r.

1.1. Clearly, if M_1, $M_2\in\{M\}$ and M_1, M_2 occur respectively as the σth and τth rows in the same matrix A_r, then $g_\sigma g_\tau^{-1}(M_2)=M_1$.

Let M_0 be a fixed element of $\{M\}$. We define inductively a sequence $C_1, C_2, \cdots$ of subsets of $\{M\}$ as follows:

$$C_1 = \{M \in \{M\} \mid \text{for some integer } r,\ M \text{ and } M_0 \text{ occur as rows in } A_r\},$$

$$C_{i+1} = \left\{M \in \{M\} \,\middle|\, \begin{matrix}\text{for some integer } r,\ M \text{ and an element}\\ \text{of } C_i \text{ occur as rows in } A_r\end{matrix}\right\}.$$

1.2. $C_1\subset C_2\subset\cdots$.

1.3. If M_1, $M_2\in C_i$ then for some g in G, $g(M_1)=M_2$.

1.4. There is an integer q such that $C_q=C_{q+i}$ for every positive integer i.

The proof of 1.2 follows directly from the definition of C_i, and 1.4 follows from 1.2 and the fact that S is finite. We prove 1.3 by induction on i. Suppose M_1, $M_2\in C_1$. Then for some r, M_1 and M_2 occur respectively as the σth and τth rows of A_r. Then $g_\tau g_\sigma^{-1}(M_1)=M_2$. Now suppose M_1, $M_2\in C_{i+1}$. Then there exist M_3, $M_4\in C_i$ and integers r, s such that M_1 and M_3 occur as rows of A_r and M_2 and M_4 occur as rows of A_s. Hence by 1.1 there exist g_1, $g_2\in G$ such that

$$g_1(M_1) = M_3,$$
$$g_2(M_2) = M_4.$$

Since M_3, $M_4\in C_i$ there is a g_3 in G such that

$$g_3(M_3) = M_4.$$

Hence $g_2^{-1}g_3g_1(M_1)=M_2$.

As a corollary we have:

1.5. If $M\in C_q$ and M occurs as a row in A_r, then each row of A_r contains an element of C_q.

1.6. Each $M\in\{M\}$ occurs as a row in the matrices A_r precisely $(k-n+1)$ times.

M contains $(n-1)$ points. For each $a\in S-M$, $M\cup a$ occurs as a row precisely once in the matrix A. Hence M occurs as a subset of precisely $(k-n+1)$ rows of A. Hence M occurs in precisely $(k-n+1)$ rows of the matrices A_r.

Now if z is the number of different elements of C_q and t is the number of matrices A_r in which elements of C_q occur, then since each A_r has $(C_{k,n})$ rows, and each $M \in C_q$ occurs as a row $(k-n+1)$ times we have

$$z(k-n+1) = t\binom{k}{n}, \qquad 0 < t \leqq n.$$

Hence

$$z(k-n+1) = \frac{t(k!)}{n!(k-n)!}$$

or

$$zn = \frac{t(k!)}{(n-1)!(k-n+1)!}.$$

But g.c.d. $(n, k!/(n-1)!(k-n+1)!) = 1$. So n divides t. Hence $t = n$, i.e., $C_q = \{M\}$, and by 1.3 G is weakly $(n-1)$-transitive over S.

DEFINITIONS.

1. $a \mid b$ means a divides b.

2. If p, x are integers then $e(p, x)$ is defined to be the greatest integer z such that $p^z \mid x$.

THEOREM 2. *For every positive integer n there is an integer $k \geqq 2n$ such that g.c.d. $(n, (C_{k,n-1})) = 1$.*

PROOF. Let $p_1, p_2, \cdots, p_r$ be the distinct prime factors of n. For each p_i let a_i be an integer such that $p_i^{a_i} > n$. Let

$$x = p_1^{a_1} \cdots p_r^{a_r}.$$

Clearly $n \mid x$. Choose $k = x+n-1$. We first note that for every integer h such that $0 < h < n$, $e(p_i, x+h) = e(p_i, h)$. For if $p^z \mid (x+h)$ then since $p_i^{a_i} \mid x$ and $h < p_i^{a_i}$, $p_i^{a_i} \nmid (x+h)$. Hence $z < a_i$. But then $p_i^z \mid x$ and thus $p_i^z \mid h$. Similarly if $p_i^z \mid h$ then $z < a_i$ and so $p_i^z \mid (x+h)$.

Thus for $i = 1, 2, \cdots, r$

$$\begin{aligned} e\left(p_i, \binom{k}{n-1}\right) &= e\left(p_i, \frac{(x+1)(x+2)\cdots(x+n-1)}{(n-1)!}\right) \\ &= \sum_{h=1}^{n-1} e(p_i, x+h) - \sum_{h=1}^{n-1} e(p_i, h) \\ &= \sum_{h=1}^{n-1} [e(p_i, (x+h)) - e(p_i, h)] \\ &= 0. \end{aligned}$$

Hence g.c.d. $(p_i, (C_{k,n-1})) = 1$. So g.c.d. $(n, (C_{k,n-1})) = 1$.

THEOREM 3. *Let $\tilde{S}$ be an infinite set, and S an infinite subset of $\tilde{S}$. Let G be weakly n-transitive over S. Then G is weakly $(n-1)$- transitive over S.*

PROOF. Let X, Y be any two $(n-1)$-subsets of S. By Theorem 2 there is an integer $k \geqq 2n$ such that $(n, (C_{k,n-1})) = 1$. Let Z be a k-subset of S such that $Z \supset X \cup Y$. Since G is weakly n-transitive over S, it is weakly n-transitive over Z. But then by Theorem 1, G is weakly $(n-1)$-transitive over Z. Hence there is a g in G such that $g(X) = Y$.

COROLLARY 1. *If S is an infinite set and G is weakly n-transitive over S then G is weakly $(n-1)$-transitive over S.*

COROLLARY 2. *If S is an infinite topological space and S is weakly n-homogeneous then S is weakly $(n-1)$-homogeneous.*

REFERENCES

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