ON THE TOPOLOGY OF TANGENT BUNDLES

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1. **Introduction.** Let M be an n-dimensional differentiable manifold. If m is a point of M then denote by $T(m)$ the tangent space to M at m. For each m in M, $T(m)$ has the algebraic structure of Euclidean n-space V^n. That is to say, if $(x_1, x_2, \cdots, x_n)$ are coordinates in a coordinate neighborhood containing the point m then the correspondence

$$(a_1, a_2, \cdots, a_n) \to a_1\left(\frac{\partial}{\partial x_1}\right)_m + a_2\left(\frac{\partial}{\partial x_2}\right)_m + \cdots + a_n\left(\frac{\partial}{\partial x_n}\right)_m$$

is a vector space isomorphism of V^n onto $T(m)$.

The tangent bundle $\mathfrak{T}(M)$ of the manifold M consists of the ordered pairs (m, v) where $m \in M$ and $v \in T(m)$. Therefore, *as a point set only*, $\mathfrak{T}(M)$ is $M \times V^n$. We shall assume that the reader is familiar with the fibre space topology which is customarily assigned to $\mathfrak{T}(M)$. (For a description of this topology and the facts concerning fibre bundles which we shall need in other parts of this paper see N. Steenrod, *The topology of fibre bundles*, Princeton, 1951.) For many manifolds M this topology differs from the topology of the product space $M \times V^n$. This fact indicates that it is the assignment of this topology which makes $\mathfrak{T}(M)$ an interesting mathematical object. It is the purpose of this paper to present an intrinsic characterization of this topology.

2. **General considerations.** Let X be a vector field defined on some open subset U of M. X assigns to each point m of U a vector $X(m)$ in $T(m)$. For each point m in M let $F(m)$ be the vector space of all real-valued functions which are of class C^1 on some neighborhood of m. If m is in U and f is in $F(m)$ then define $\langle X(m), f\rangle$ to be the value of the differential df at the vector $X(m)$.

Let $m_0 \in U$ and $f \in F(m_0)$. Then there is a neighborhood V of m_0 which is contained in U such that $\langle X(m), f\rangle$ is defined on V. $\langle X, f\rangle$ is then a real-valued function on V.

DEFINITION. The vector field X is said to be *continuous at the point* m_0 if $\langle X, f\rangle$ is continuous on a neighborhood of m_0 for each f in

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$F(m_0)$. X is said to be *continuous on* U of it is continuous at each point of U.

Let $\pi: \mathfrak{I}(M) \to M$ be the projection defined by $\pi(m, v) = m$. If X is a vector field on an open subset U of M we may define $\chi: U \to \mathfrak{I}(M)$ by $\chi(m) = (m, X(m))$. The correspondence χ then satisfies $\pi\chi =$ identity on U. The vector fields on U are in one-to-one correspondence with the mappings χ of U into $\mathfrak{I}(M)$ which satisfy $\pi\chi =$ identity.

The relationship between our definition of continuous vector field and the problem of characterizing the topology in $\mathfrak{I}(M)$ is indicated by the following lemma.

LEMMA. *A vector field X defined on a coordinate neighborhood U of M is continuous on U if and only if the associated mapping χ is continuous in the usual topology on $\mathfrak{I}(M)$.*

PROOF. Let $(x_1, x_2, \cdots, x_n)$ be coordinates in U. Let $X = v_1\partial/\partial x_1 + v_2\partial/\partial x_2 + \cdots + v_n\partial/\partial x_n$ where $v_1, v_2, \cdots, v_n$ are functions on U. If $m \in U$ and $f \in F(m)$ then $df = w_1dx_1 + w_2dx_2 + \cdots + w_ndx_n$ where $w_1, w_2, \cdots, w_n$ are *continuous* functions on some neighborhood V of m which is contained in U. Hence $\langle X, f\rangle = v_1w_1 + v_2w_2 + \cdots + v_nw_n$ on V.

Let $p_j(x_1, x_2, \cdots, x_n) = x_j$ for $j = 1, 2, \cdots, n$. If we assume that X is continuous then $v_j = \langle X, p_j\rangle$ is continuous on U for $j = 1, 2, \cdots, n$, and so χ is continuous in the usual topology on $\mathfrak{I}(M)$. Conversely, if we assume that χ is continuous in the usual topology on $\mathfrak{I}(M)$ then $v_1, v_2, \cdots, v_n$ are continuous, and hence $\langle X, f\rangle$ is continuous on V. Since this is true for every $m \in U$ and $f \in F(m)$ we find that X is continuous on U.

3. **The characterization of the topology of** $\mathfrak{I}(M)$. In order to distinguish the two topologies which we shall be considering on the same point set $\mathfrak{I}(M)$ we shall refer to the usual fibre bundle topology as the o-topology on $\mathfrak{I}(M)$. The actual topology of $\mathfrak{I}(M)$ will be called the $*$-topology. The five conditions which we shall assume for the $*$-topology are the following:

(i) $\mathfrak{I}(M)$ is a regular space.

(ii) Each point of $\mathfrak{I}(M)$ has an open neighborhood in the $*$-topology whose closure in the o-topology is compact in the o-topology.

(iii) The projection π is a continuous mapping.

(iv) For each point m in M the topology induced on $\pi^{-1}(m)$ by the $*$-topology is the same as that induced by the o-topology.

(v) If X is any continuous vector field on an open set U in M then the associated mapping χ is a continuous mapping of U into $\mathfrak{I}(M)$.

THEOREM. *If the $*$-topology satisfies conditions* (i)–(v) *then it is the same as the* o*-topology.*

PROOF. If U is any coordinate neighborhood of M and $(x_1, x_2, \cdots, x_n)$ are coordinates in U then we may associate with these a one-to-one mapping h of $U \times V^n$ onto $\pi^{-1}(U)$ defined by

$$h(m, (a_1, a_2, \cdots, a_n)) = \left(m, a_1\left(\frac{\partial}{\partial x_1}\right)_m + a_2\left(\frac{\partial}{\partial x_2}\right)_m + \cdots + a_n\left(\frac{\partial}{\partial x_n}\right)_m\right).$$

Now suppose that under the assumptions (i)–(v) we may prove that for every coordinate neighborhood U the corresponding mapping h is a homeomorphism. Let U and U' be two overlapping coordinate neighborhoods and let h and h' be their corresponding mappings, respectively. Define $g = h^{-1}h'$ on $(U \cap U') \times V^n$. Then $g(m, (a_1, a_2, \cdots, a_n)) = (m, l(m)(a_1, a_2, \cdots, a_n))$ where $l(m)$ is a nonsingular linear transformation of V^n, that is, an element of $GL(n)$. The correspondence $m \rightarrow l(m)$ is a continuous mapping from $U \cap U'$ into $GL(n)$. Consequently $\mathfrak{J}(M)$ has the structure of a coordinate bundle in which the $l(m)$'s are the coordinate transformations and its topology is the familiar o-topology. The proof of the theorem is therefore reduced to the demonstration that the mapping h is a homeomorphism. We remind the reader that $\pi^{-1}(U)$ has the topology induced by the $*$-topology in $\mathfrak{J}(M)$ satisfying conditions (i)–(v).

We first prove that h is continuous. Let V^* be an open set in $\pi^{-1}(U)$ and let (m_0, v_0) be a point in V^*. The vector field X for which $X(m) = v_0$ for all m in U is continuous on U. By (iii) and (v) the associated mapping χ, for which $\chi(m) = (m, v_0)$, is a homeomorphism of U into $\mathfrak{J}(M)$. $\chi(U) \cap V^*$ is open in the topology induced on $\chi(U)$ so $O = \pi(\chi(U) \cap V^*)$ is an open subset of U. Since $(m_0, v_0) \in \chi(U)$ we have $m_0 \in O$.

For each $m \in O$ let $W_m^* = \pi^{-1}(m) - (\pi^{-1}(m) \cap V^*)$ and let $\lambda(m)$ be the infimum of the lengths of all vectors $w - w_0$ where $h(m, w) \in W_m^*$ and $h(m, w_0) = (m, v_0)$. We shall prove that there is a neighborhood N of m_0, $N \subset O$, and a constant $\epsilon > 0$ such that $\lambda(m) \geqq \epsilon$ for $m \in N$. This will complete the proof of the continuity of h since we may then construct the neighborhood $N \times W$ where

$$W = \{w \mid w \in V^n, \text{ length of } w - w_0 < \epsilon\}.$$

$N \times W$ is a neighborhood of (m_0, w_0) and $h(N \times W) \subset V^*$.

Assume that no such neighborhood N and constant ϵ exist. Then there is a Cauchy sequence (m_1, w_1), (m_2, w_2), $\cdots$ of points of $O \times V^n$ with $h(m_i, w_i) \in W^*_{m_i}$ which converges to (m_0, w_0). If we suppose that there are an infinite number of distinct points among $m_1, m_2, \cdots$ then by passing to a subsequence we may assume that $m_1, m_2, \cdots$ are themselves distinct. Let $F = \{m_1, m_2, \cdots\}$. Then $F \cup \{m_0\}$ is a closed set. The correspondence $m_i \rightarrow w_i$, $i = 0, 1, 2, \cdots$, is a section of $U \times V^n$ over $F \cup \{m_0\}$. Since V^n is a solid space this section over $F \cup \{m_0\}$ may be extended to a section ϕ of $U \times V^n$ over U. The mapping $\chi = h\phi$ of U into $\mathfrak{Z}(M)$ defines a vector field which is continuous on U. By (v) the mapping χ is a homeomorphism of U into $\mathfrak{Z}(M)$. Thus $\chi(m_0)$ must be a limit point of $\chi(F)$. But $\chi(m_i) = h\phi(m_i) = h(m_i, w_i) \notin V^*$ for $i = 1, 2, \cdots$, and this means that $\chi(m_0) = (m_0, v_0)$ is not in V^* since V^* is open. Since this is not true we conclude that there are only a finite number of points among $m_1, m_2, \cdots$.

By passing to a subsequence we may assume that $m_1 = m_2 = \cdots = m_0$ and hence that $h(m_1, w_1)$, $h(m_2, w_2)$, $\cdots$ are all in $\pi^{-1}(m_0)$. By (iv) the sequence $h(m_0, w_1)$, $h(m_0, w_2)$, $\cdots$ converges to $h(m_0, w_0) = (m_0, v_0)$ in $\pi^{-1}(m_0)$. But $h(m_i, w_i) \notin V^*$ for $i = 1, 2, \cdots$, so again we find that $(m_0, v_0) \notin V^*$ which is not true. This completes the proof that h is continuous.

There remains the demonstration that h^{-1} is continuous. Let W be an open set in $U \times V^n$, let $(m_0, w_0) \in W$, and let $h(m_0, w_0) = (m_0, v_0)$. Suppose that for every open subset V^* of $\pi^{-1}(U)$ which contains (m_0, v_0) we find that $h^{-1}(V^*)$ is not contained in W. Let

$$C^* = \{z^* \mid z^* \in \pi^{-1}(m_0),\ z^* \notin h(W)\}.$$

Since $h^{-1}(C^*)$ is closed as a subset of the space $(\pi h)^{-1}(m_0)$ so by (iv) C^* is closed as a subset of the space $\pi^{-1}(m_0)$. By (iii) $\pi^{-1}(m_0)$ is a closed set so C^* is a closed set in $\pi^{-1}(U)$. We shall show that the point (m_0, v_0) and the closed set C^* cannot be separated by disjoint neighborhoods. This will contradict condition (i) and thereby prove that h^{-1} is continuous.

By (ii) there is an open set B^* containing (m_0, v_0) such that the closure of $h^{-1}(B^*)$ is compact. Let V^* be any open set containing (m_0, v_0) and let $N_1 \supset N_2 \supset \cdots$ be a nested sequence of open neighborhoods of m_0 in M such that the diameter of $N_j \rightarrow 0$ as $j \rightarrow \infty$. By (iii) $V^*_j = B^* \cap \pi^{-1}(N_j) \cap V^*$ is an open set containing (m_0, v_0), and by our assumption $h^{-1}(V^*_j)$ contains a point (m_j, w_j) such that $(m_j, w_j) \notin W$. Thus (m_1, w_1), (m_2, w_2), $\cdots$ is a sequence of points in the compact closure of $h^{-1}(B^*)$ with $m_j \rightarrow m_0$ as $j \rightarrow \infty$. By passing to a subsequence we may assume that (m_1, w_1), (m_2, w_2), $\cdots$ is a Cauchy sequence. Let (m_0, w_∞) be the limit point. Since W is an open set $(m_0, w_\infty) \notin W$. On the other hand, (m_1, w_1), (m_2, w_2), $\cdots$ are all in $h^{-1}(V^*)$ so that (m_0, w_∞) is in the closure of $h^{-1}(V^*)$. Since h is continuous $h(m_0, w_\infty)$ is in the closure of V^*. We have thus proved that the closure of any open set of $\pi^{-1}(U)$ which contains (m_0, v_0) must contain a point of C^*. It follows that C^* and (m_0, v_0) cannot be separated by disjoint neighborhoods and the proof of the continuity of h^{-1} is complete.

4. **An example.** In this section we shall construct an example which illustrates the essential character of condition (ii), for it produces a $*$-topology on $\mathfrak{I}(M)$ which satisfies conditions (iii), (iv), and (v) and is even normal, thereby satisfying condition (i), and yet it fails to satisfy condition (ii) and is therefore not the o-topology.

Let $\mathfrak{V}$ be Euclidean 3-space and let $\mathfrak{D}$ be the discrete subgroup of $\mathfrak{V}$ generated by the vectors (0, 1, 0) and (0, 0, 1). The quotient group $\mathfrak{M} = \mathfrak{V}/\mathfrak{D}$ is then a 3-dimensional manifold. Let α be a fixed irrational real number, $0 < \alpha < 1$. Let $\mathcal{P}$ be the 2-dimensional subspace of $\mathfrak{V}$ generated by the vectors (1, 0, 0) and $(0, 1, \alpha)$. Let $\mathfrak{I}$ be the subset of $\mathfrak{M}$ consisting of those points which are represented by elements of $\mathcal{P}$. Since no two distinct points of $\mathcal{P}$ can represent the same element of $\mathfrak{M}$ the natural mapping of $\mathfrak{V}$ onto $\mathfrak{M}$ induces a one-to-one correspondence h between the points of $\mathcal{P}$ and those of $\mathfrak{I}$. h is a continuous mapping of $\mathcal{P}$ onto $\mathfrak{I}$.

Let M be the 1-dimensional subspace of $\mathcal{P}$ generated by the vector (1, 0, 0). M is the real line, a one-dimensional manifold. The point $(x, y, \alpha y)$ of $\mathcal{P}$ will be denoted simply by (x, y). $\mathcal{P}$ has the topology of Euclidean 2-space and is a model for $\mathfrak{I}(M)$ with the o-topology. The point $h(x, y, \alpha y)$ of $\mathfrak{I}$ will be denoted by $h(x, y)$. The topology induced on $\mathfrak{I}$ by the topology in $\mathfrak{M}$ will be called the †-topology.

If $(x_0, y_0) \in \mathcal{P}$ let

$$S_\epsilon(x_0, y_0) = \left\{(x, y, z) \,\middle|\, (x - x_0)^2 + (y - y_0)^2 + (z - \alpha y_0)^2 < \epsilon^2\right\}.$$

The sets $S_\epsilon(x_0, y_0) \cap \mathcal{P}$ are a basis for the o-topology on $\mathcal{P}$. On the other hand, the sets

$$U_\epsilon(x_0, y_0) = h(S_\epsilon(x_0, y_0)) \cap \mathfrak{I}$$

are a basis for the †-topology on $\mathfrak{I}$. Let

$$F(x_0) = \left\{(x_0, y) \,\middle|\, -\infty < y < \infty\right\}.$$

Let

$$V_\epsilon(x_0, y_0) = \left[U_\epsilon(x_0, y_0) - h(F(x_0))\right] \cup h\left[F(x_0) \cap S_\epsilon(x_0, y_0)\right].$$

We take the family of all sets $U_\epsilon(x_0, y_0)$ and $V_\epsilon(x_0, y_0)$ as the basis for the $*$-topology in $\mathfrak{J}$. Note that h is still a continuous mapping of $\mathcal{P}$ onto $\mathfrak{J}$ with this topology.

The projection π is defined by $\pi[h(x, y)] = x$. π is continuous in the $*$-topology so condition (iii) is satisfied. Furthermore, due to the presence of the sets $V_\epsilon(x_0, y_0)$, the topology induced on $h(F(x_0)) = \pi^{-1}(x_0)$ by the $*$-topology is the same as that induced on $F(x_0)$ by the $\circ$-topology, that is, condition (iv) is satisfied.

Let $X(x)$ be a vector field on M. Then $X(x_0) = y_0(\partial/\partial x)_{x_0}$ for each $x_0 \in M$ and we define $\phi(x_0) = (x_0, y_0)$. ϕ is then a mapping of M into $\mathcal{P}$. If $X(x)$ is continuous on M then ϕ is continuous and hence $\chi = h\phi$ is a continuous mapping of M into $\mathfrak{J}$. But χ is the mapping of M into $\mathfrak{J}$ associated with $X(x)$ and we see that condition (v) is satisfied.

Let C^* be a closed set in $\mathfrak{J}$ and let p be a point of $\mathfrak{J}$ which is not in C^*. Then C^* is also closed in the $\dagger$-topology. Thus $C^* = C \cap \mathfrak{J}$ where C is a closed subset of the manifold $\mathfrak{M}$ and $p \notin C$. Since $\mathfrak{M}$ is normal we may find disjoint open sets U and V in $\mathfrak{M}$ such that $p \in U$ and $C \subset V$. Let $U^* = U \cap \mathfrak{J}$ and $V^* = V \cap \mathfrak{J}$. Then U^* and V^* are disjoint open sets in the $\dagger$-topology and hence also in the $*$-topology. Since $p \in U^*$ and $C^* \subset V^*$ we see that $\mathfrak{J}$ is normal in the $*$-topology. On the other hand, each set $V_\epsilon(x_0, y_0)$ or $U_\epsilon(x_0, y_0)$ contains points $h(x, y)$ such that (x, y) is arbitrarily distant from $(0, 0)$ in the $\circ$-topology. Consequently condition (ii) is violated at each point of T.

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