DISCONJUGACY OF SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS WITH NON-NEGATIVE COEFFICIENTS

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Introduction. In a recent paper [5] Zeev Nehari established a criterion for disconjugacy[1] of

$$(p(x)y')' + f(x)y = 0, \qquad a \leqq x < \infty,$$

for $p(x) \equiv 1$ and $f(x)$ positive and continuous, in terms of the least eigenvalue $\lambda(b)$ of the corresponding eigenvalue problem

$$(p(x)y')' + \lambda f(x)y = 0, \quad y(a) = y'(b) = 0, \quad a \leqq x \leqq b < \infty.$$

His basic criterion for disconjugacy turned out to be $1 < \lambda(b)$ for $a < b < \infty$. If $p(x) > 0$ is continuous and $\int_a^\infty 1/p = \infty$, then the substitution $t = \int_a^x 1/p$ transforms the first equation into one where $p \equiv 1$ and the new t-interval is infinite. Thus this more general case reduces to Nehari's and again the criterion is $1 < \lambda(b)$. However part of the whole picture seems to be lost by this transformation. If $\int_a^\infty 1/p = \infty$ and $\int_a^\infty f = \infty$, then the well-known Leighton-Wintner Oscillation Theorem [4; 8] states that the equation $(py')' + fy = 0$ is oscillatory.[2] Also, if both integrals are finite, the equation is nonoscillatory.[3] These conclusions may be strengthened to *strongly oscillatory* and *strongly nonoscillatory*, respectively, as is evident from the definitions [5]. The cases of interest in this paper are those where one integral is infinite and one is finite.

As will be seen in §5, if $\int_a^\infty f = \infty$ and $(py')' + fy = 0$ is disconjugate, then $\lambda(b) \to 0$ (as the reciprocal of $\int_a^b f$) and another disconjugacy criterion is needed for this case.

However, by noting that the necessary conditions for disconjugacy are related to quadratic functionals, which Reid [6] has connected to the nonexistence of focal points, Nehari's and other necessary conditions are obtained for more general conditions on the coefficients. Also, the case of $\int_a^\infty f = \infty$, together with disconjugacy of (1), is discussed.

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[1] Every nontrivial solution has at most one zero on $a \leqq x < \infty$.

[2] Every solution has infinitely many zeros on $a \leqq x < \infty$.

[3] Every nontrivial solution has at most a finite number of zeros on $a \leqq x < \infty$.

1. Variational criteria for nonexistence of focal points. Consider the real differential equation

$$L[y] = (p(x)y')' + f(x)y = 0 \qquad \text{on } a \leqq x < \infty, \tag{1}$$

where, almost everywhere (a.e.) on $a \leqq x < \infty : p(x) > 0$, $f(x) \geqq 0$; for each b, $a < b < \infty$, $p(x)$, $p^{-1}(x)$, and $f(x)$ are Lebesgue integrable $(L(a, b))$ on $a \leqq x \leqq b$ and $\int_b^\infty f > 0$. The designation (1) will stand for $L[y] = 0$, a.e. on $a \leqq x < \infty$, together with the above-mentioned conditions on the coefficients. It is understood that all coefficients are defined a.e. on the interval considered.

A function $y(x)$ is said to be a *solution* of (1) provided $y(x)$ is *absolutely continuous* (a.c.) on $a \leqq x \leqq b$ for $a < b < \infty$ and there exists an *associated* absolutely continuous function $y_1(x)$ on $a \leqq x \leqq b$ such that $y_1(x) = p(x)y'(x)$ a.e. on $a \leqq x \leqq b$ for $a < b < \infty$ and $y_1'(x) = -f(x)y(x)$ a.e. on $a \leqq x < \infty$. The subscript 1 will be used to denote this *associated function*. Thus, the fundamental differential system is[4]

$$y' = p^{-1}(x)y_1, \qquad y_1' = -f(x)y. \tag{1'}$$

A point b is said to be a *right focal point of* a (with respect to (1)) provided $a < b < \infty$ and there exists a nontrivial solution $y(x)$ of (1) (in the sense of the preceding paragraph) such that it, and its associated function, satisfy the two point boundary conditions:

$$y(b) = y_1(a) = 0. \tag{2}$$

Note that if $p(x)$ is continuous and $p(a) > 0$, then (2) becomes the familiar focal point conditions: $y(b) = y'(a) = 0$.

Left focal points are defined similarly only with $b < a$.

Recently W. T. Reid [6] pointed out a criterion for nonexistence of focal points in terms of the quadratic functional

$$\mathrm{I}[y] = \int_a^b (p(y')^2 - fy^2). \tag{3}$$

Include in the designation (3), the same conditions on the coefficients as for (1). Note that (1) is the *Euler equation* of (3).

The following statements aid in the formulation of Reid's criterion:

(F_+) $a < x \leqq b$ contains no (right) focal point of a.

(F_-) $a \leqq x < b$ contains no (left) focal point of b.

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[4] Throughout this paper the equality sign (=) between two functions will denote that each function is defined a.e. and that they are equal a.e. on the interval considered. Of course, if both functions exist everywhere and are continuous on the interval then this "equality" is equivalent to identity. The same convention is observed for inequalities. The meaning should be clear from the content.

The following basic theorem is that of Reid's [6, p. 198] stated in the more general terms of the conjugate point discussion of [7], but only for the real scalar case.

THEOREM R. *In order for* (F_+) *to be true it is necessary and sufficient that* $I[y]>0$, *for all nontrivial functions* $y(x)$ *belonging to the class* I_+: $\{y(x)$ *a.c.*, $y(b)=0$, $I[y]$ *exists and is finite*$\}$. *This statement is also true with* (F_+) *replaced by* (F_-), I_+ *by* I_- *and* "$y(b)=0$" *by* "$y(a)=0$."

2. **Disconjugacy and nonexistence of focal points when** $\int^\infty 1/p=\infty$. A slight extension of the well-known Prüfer transformation [1; 2, Chapter 8] shows that for each nontrivial solution $y(x)$ of (1), there exist absolutely continuous functions $r(x)$ and $\theta(x)$ such that for $a\leqq x\leqq b<\infty$,

$$y(x) = r(x)\sin\theta(x), \qquad p(x)y'(x) = r(x)\cos\theta(x) \tag{4}$$

and, furthermore,

$$r'(x) = r(x)[p^{-1}(x) - f(x)]\frac{\sin 2\theta(x)}{2}, \qquad r(a) > 0. \tag{5a}$$

$$\theta'(x) = p^{-1}(x)\cos^2\theta(x) + f(x)\sin^2\theta(x) \geqq 0, \qquad 0 \leqq \theta(a) < \pi. \tag{5b}$$

This transformation will be used to establish necessary conditions for disconjugacy of (1) on $a\leqq x<\infty$. The first lemma is due to Hille [3] for $p\equiv 1$ and continuous $f(x)$; however, he should have had an additional condition on $f(x)$ so that it would not be identically zero for large x, as is illustrated by the following:

EXAMPLE 2.1. Let x_0 be the smallest positive zero of the solution $Y(x)$ of $y''+xy=0$, $y(0)=1$, $y'(0)=0$. Let $p(x)\equiv 1$, $f(x)=-x$ on $a=-x_0\leqq x<0$ and $f(x)=0$ on $0\leqq x<\infty$. Then clearly (1) has a solution satisfying $y(a)=0$, $y'(a)>0$ but $y'(x)=0$ on $0\leqq x<\infty$. Thus $y(x)$ is only *nondecreasing* on $a\leqq x<\infty$.

LEMMA 2.1. *Let* (1) *be disconjugate on* $a\leqq x<\infty$ *and, in addition, require that*

$$\int_a^\infty p^{-1} = \lim_{b\to\infty}\int_a^b p^{-1} = \infty. \tag{i}$$

If $y=u(x)$ *is a solution of* (1) *and the one-point boundary conditions*:

(6) $y(a) = 0, \quad y_1(a) > 0$ ($y'(a) > 0$, *if both it and* $p(a) > 0$ *exist*),

then $u_1(x)>0$ (*and hence* $u'(x)>0$) *and* $u(x)$ *is an increasing function on* $a\leqq x<\infty$.

PROOF. Perform the Prüfer transformation (4):

$$u(x) = r(x) \sin \theta(x), \qquad p(x)u'(x) = u_1(x) = r(x) \cos \theta(x).$$

Then $u(x)$, $u_1(x)$, $r(x)$ and $\theta(x)$ are all absolutely continuous on $a \leqq x \leqq b$, $a < b < \infty$, and (5b) becomes

$$\theta' = p^{-1}(x) \cos^2 \theta + f(x) \sin^2 \theta, \qquad \theta(a) = 0,$$

or

$$\theta(x) = \int_a^x (p^{-1} \cos^2 \theta + f \sin^2 \theta) \qquad (\text{nondecreasing on } a \leqq x < \infty).$$

Also, since (1) is disconjugate and $p^{-1} > 0$, then $0 < \theta(x) < \pi$ for $a < x < \infty$. Assume that $a < c < \infty$ and $\theta(c) = \pi/2$. If $c < c_1$ and $\theta(x) = \pi/2$ on $c \leqq x \leqq c_1$, then $\theta'(x) = f(x) = 0$ on $c \leqq x \leqq c_1$. Since $\int_c^\infty f > 0$, there exists c_2, $c < c_2 < \infty$, such that $\pi/2 < \theta(c_2) \leqq \theta(x) < \pi$ on $c_2 \leqq x < \infty$. Now

$$\theta(x) - \theta(c_2) \geqq \int_{c_2}^x p^{-1} \cos^2 \theta \geqq \cos^2 \theta(c_2) \int_{c_2}^x p^{-1} \to \infty, \text{ as } x \to \infty,$$

which gives a contradiction and, hence, $0 < \theta(x) < \pi/2$ for $a < x < \infty$. Finally, $u_1(x) > 0$ and $u'(x) > 0$ on $a < x < \infty$, completing the proof of the lemma. Note also that $\theta(x) \to \pi/2$ as $x \to \infty$, and $\int_a^\infty f < \infty$.

It follows immediately that for $a < b < \infty$, there exists no (left) focal point of b on $a \leqq x < b$ and Theorem R applies, giving:

LEMMA 2.2. *If* (1) *is disconjugate on* $a \leqq x < \infty$ *and* (i) $\int_a^\infty p^{-1} = \infty$, *then* $I[y] = \int_a^b [p(y')^2 - fy^2] > 0$ *for all* $y(x) \not\equiv 0$ *in class* I_-.

It is easily seen that this is equivalent to Nehari's assertion of $1 < \lambda(b)$ and the corresponding Wirtinger's inequality, for $p \equiv 1$ and $f(x)$ positive and continuous.

Following Nehari [**5**], let $a < x_0 < b < \infty$, $0 \leqq \alpha < 1 < \beta < \infty$, and

$$y_0(x) = \begin{cases} \left(\int_a^x p^{-1} \Big/ \int_a^{x_0} p^{-1} \right)^{\beta/2} & \text{on } a \leqq x \leqq x_0, \\ \left(\int_a^x p^{-1} \Big/ \int_a^{x_0} p^{-1} \right)^{\alpha/2} & \text{on } x_0 \leqq x \leqq b. \end{cases}$$

Certainly, $y_0(x)$ belongs to class I_- and substitution in $I[y] > 0$ yields a slight extension of Nehari's inequality:

THEOREM 2.1. *If* (1) *is disconjugate*, (i) $\int_a^\infty p^{-1} = \infty$, *and* $0 \leqq \alpha < 1 < \beta$, *then on* $a \leqq x < \infty$:

$$\left(\int_a^t p^{-1}\right)^{1-\beta}\int_a^t f(t)\left(\int_a^x p^{-1}\right)^{\beta}dt$$

$$(7) \qquad + \left(\int_a^x p^{-1}\right)^{1-\alpha}\int_x^\infty f(t)\left(\int_a^t p^{-1}\right)^{\alpha}dt$$

$$\leqq \frac{\beta-\alpha}{4}\left[1+\frac{1}{(\beta-1)(1-\alpha)}\right] < \infty;$$

and the special cases (see [4]):

COROLLARY 2.1.1. (Hille [3]). $\alpha=0$, $\beta=2$ *yields* $\int_a^x p^{-1}\int_x^\infty f<1$ *for* $a\leqq x<\infty$, *as a necessary condition for disconjugacy of* (1) *when* $\int^\infty p^{-1}=\infty$;

COROLLARY 2.1.2. $0\leqq\alpha<1$, $\beta=2$: $(\int_a^x p^{-1})^{1-\alpha}\int_x^\infty f(t)(\int_a^t p^{-1})^\alpha dt \leqq 1/(1-\alpha)<\infty$ *and* $\int_a^\infty f(t)(\int_a^t 1/p)^\alpha dt<\infty$ *are necessary disconjugacy conditions for* (1) *when* $\int^\infty p^{-1}=\infty$.

This section is concluded with an example which illustrates the results of the section and will also be used in the succeeding discussion.

EXAMPLE 2.2. Let $p(x)=f(x)=e^{-2x}$ on $a=0\leqq x<\infty$; then $\int^\infty p^{-1}=\infty$, $\int^\infty f<\infty$, (1) is disconjugate, and $y=u(x)=xe^x$ illustrates Lemma 2.1.

3. Disconjugacy when $\int_a^\infty f=\infty$.

Consider first an example related to the previous one.

EXAMPLE 3.1. Let $p(x)=f(x)=e^{2x}$ on $a=0\leqq x<\infty$; then $\int^\infty p^{-1}<\infty$ and $\int^\infty f=\infty$. Note that $y=xe^{-x}$ is a solution of (1) and the initial conditions $y(a)=0$, $y'(a)>0$.

This solution $y=xe^{-x}$ suggests the following general result:

LEMMA 3.1. *Let* (1) *be disconjugate on* $a\leqq x<\infty$ *and, in addition,*

$$\text{(ii)} \qquad \int_a^\infty f = \lim_{b\to\infty}\int_a^b f = \infty.$$

If $y=v(x)$ *is a solution of* (1) *and the boundary conditions* (6), *then there exist numbers* c_1, c_2 *such that* $a<c_1\leqq c_2<\infty$ *and*

$$v'(x)\begin{cases} >0 & \text{on } a\leqq x<c_1,\\ =0 & \text{on } c_1\leqq x\leqq c_2,\\ <0 & \text{on } c_2<x<\infty.\end{cases}$$

Also $c_1=c_2$, *if* $f(x)>0$ *on* $a\leqq x<\infty$.

PROOF. As for Lemma 2.2, use the Prüfer Transformation: $v(x) = r(x)\sin\theta(x)$ and $p(x)v'(x) = r(x)\cos\theta(x)$ and consider

$$\theta' = p^{-1}\cos^2\theta + f\sin^2\theta, \qquad \theta(a) = 0. \tag{5b}$$

Thus $0 < \theta(x) < \pi$ for $a < x < \infty$. Assume $\theta(x) < \pi/2$ for $a \leqq x < \infty$ and $a < a_1 < \infty$. Then

$$\theta(x) - \theta(a_1) \geqq \sin^2\theta(a_1)\int_{a_1}^{x} f \to \infty, \qquad \text{as } x \to \infty,$$

which contradicts disconjugacy of (1). Thus, there exists a (first) number c_1, $a < c_1 < \infty$, such that $\theta(c_1) = \pi/2$. Also, since $\int_{c_1}^{\infty} f > 0$ there exists a (largest) number c_2, $c_1 \leqq c_2 < \infty$, such that $\theta(x) = \pi/2$ on $c_1 \leqq x \leqq c_2$. Hence, on $c_2 < x < \infty$, $\pi/2 < \theta(x) < \pi$ and, in fact, $\theta(x) \to \pi$ as $x \to \infty$. Note that $c_1 = c_2$, if $f(x) > 0$ on $a < x < \infty$. The conclusion of the lemma follows immediately.

Note that there exists no (right) focal point of $c = c_2$ on $c < x < \infty$ and, therefore, by Theorem R for $c_2 \leqq c < b < \infty$,

$$I[y] = \int_c^b (p(y')^2 - fy)^2 > 0 \text{ for } y \not\equiv 0 \text{ in class } I_+ \text{ on } c \leqq x \leqq b.$$

For class I_+, $y(b) = 0$, let $c < x_0 < b < \infty$, $0 \leqq \alpha < 1 < \beta < \infty$ and

$$y_1(x) = \begin{cases} \left(\int_x^b p^{-1} \Big/ \int_{x_0}^b p^{-1}\right)^{\alpha/2} & \text{on } c \leqq x \leqq x_0, \\ \left(\int_x^b p^{-1} \Big/ \int_{x_0}^b p^{-1}\right)^{\beta/2} & \text{on } x_0 \leqq x \leqq b, \end{cases}$$

and, as before,

THEOREM 3.1. *If* (1) *is disconjugate,* (ii) $\int_a^\infty f = \infty$, *and* $0 \leqq \alpha < 1 < \beta < \infty$, *then there exists a number* c, *such that* $a < c < \infty$ *and for* $c < x_0 < \infty$:

$$\begin{aligned} &\left(\int_{x_0}^{\infty} p^{-1}\right)^{1-\alpha} \int_c^{x_0} f(t)\left(\int_t^{\infty} p^{-1}\right)^{\alpha} dt \\ &\qquad + \left(\int_{x_0}^{\infty} p^{-1}\right)^{1-\beta} \int_{x_0}^{\infty} f(t)\left(\int_t^{\infty} p^{-1}\right)^{\beta} \\ &\qquad \leqq \frac{\beta-\alpha}{4}\left[1 + \frac{1}{(\beta-1)(1-\alpha)}\right] < \infty. \end{aligned} \tag{8}$$

COROLLARY 3.1.1. *Let* $\alpha = 0$, $\beta = 2$, *then* $\int_x^\infty p^{-1}\int_c^x f < 1$ *on* $c \leqq x < \infty$ *is a necessary condition for disconjugacy of* (1) *when* $\int^\infty f = \infty$.

Note that this last result is the same as that of Corollary 2.1.1 with p^{-1} and f interchanged. However, the more general inequalities (7) and (8) appear to be quite different. The next section will show that p^{-1} and f can be interchanged in (7) and (8) when more restrictions are placed on f.

4. **The "reciprocal" equation.** Recall the first order system (1′) $y' = p^{-1}(x)z$, $z' = -f(x)y$ on $a \leqq x \leqq b < \infty$. If $y(x)$ is a solution of (1), then there exists a corresponding a.c. function $y_1(x)$ such that $(y(x), y_1(x))$ is a solution of (1′). Furthermore, if $f(x) > 0$ then $y_1(x)$ is a solution of the "reciprocal" equation

$$(9) \qquad (p^*(x)z')' + f^*(x)z = 0, \qquad p^* = f^{-1}, \qquad f^* = p^{-1}.$$

Note that if $\int^\infty p^{-1} = \infty$ and $\int^\infty f < \infty$, then $\int^\infty p^{*-1} < \infty$ and $\int^\infty f^* = \infty$, and conversely. Examples (2.1) and (3.1) illustrate (1) and (9). In addition to the hypotheses in §1, assume that $f^{-1}(x)$ belongs to $L(a, b)$ for $a < b < \infty$.

LEMMA 4.1. *If* $\int^\infty p^{-1} = \infty$, $f(x) > 0$ *and* (1) *is disconjugate on* $a \leqq x < \infty$, *then* (9) *is disconjugate.*

PROOF. Let $y = u(x)$ of Lemma 2.1 and $u_1(x) = p(x)u'(x)$ where $u_1(x)$ is a.c. Then $u_1(x) > 0$ on $a \leqq x < \infty$, is a solution of (9) and, hence, (9) is disconjugate by the Sturm separation theorem. Unfortunately, the converse is not as strong.

LEMMA 4.2. *If* $\int^\infty p^{-1} = \infty$, $f(x) > 0$ *and* (9) *is disconjugate on* $a \leqq x < \infty$, *then any solution of* (1) *can have at most two zeros on*

$$a \leqq x < \infty.$$

PROOF. Let $z = v(x)$ of Lemma 3.1. Then there exists an a.c. function $y(x) = p(x)v'(x)$ which is a solution of (1). Thus $y(x)$ has one zero on $a < x < \infty$ and, hence, no solution can have more than two zeros there.

The following example shows that the latter result can not be improved to correspond to Lemma 4.1.

EXAMPLE 4.1. Let $p(x) = f(x) = e^{-2|x|}$, $a = -1 \leqq x < \infty$. Then $\int^\infty p^{-1} = \infty$ and $y(x) = (|x| - 1)e^{|x|}$ is a solution of (1) with *two zeros* at $x = \pm 1$. Also,

$$z(x) = \begin{cases} (1 + x)e^x, & -1 \leqq x \leqq 0 \\ (3x + 1)e^{-x}, & 0 \leqq x < \infty \end{cases} > 0 \qquad \text{on } a < x < \infty$$

is a solution of (9), making (9) disconjugate on $a < x < \infty$.

Lemmas 4.1 and 4.2 may now be applied to Theorems 2.1 and 3.1 giving:

THEOREM 4.1. *If* (1) *is disconjugate,* (ii) $\int_a^\infty f=\infty$, $f(x)>0$ *and* $0\leqq\alpha<1<\beta$, *then the necessary inequality* (7) *is true with* p^{-1} *and* f *interchanged and* a *replaced by some* $c>a$.

THEOREM 4.2. *If* (1) *is disconjugate,* (i) $\int_a^\infty p^{-1}=\infty$, $f(x)>0$ *and* $0\leqq\alpha<1<\beta$, *then* (8) *is true with* p^{-1} *and* f *interchanged and* c *replaced by* a.

The author has not succeeded in proving or disproving either of these theorems when the condition "$f(x)>0$ a.e." is removed, although they are true for the special case $\alpha=0$ and $\beta=2$, as is shown by Corollaries 2.1.1 and 3.1.1.

5. **Growth of an eigenvalue.** Recall from the introduction that Nehari [5] gives a criterion for disconjugacy of (1) in terms of the least eigenvalue $\lambda(b)$ of an associated boundary value problem when $p\equiv 1$ and, hence, for $\int^\infty 1/p=\infty$. A natural question is raised then as to the bounds on $\lambda(b)$ when $\int^\infty f=\infty$ and (1) is disconjugate. In this section necessary conditions on $\lambda(b)$ are obtained without the assumption of disconjugacy (but only if $\int^\infty 1/p<\infty$) and, hence, are not sufficient for disconjugacy of (1).

The discussion of this section will be based on the assumptions (additional to those with (1)):

$$\text{(ii)} \qquad \int^\infty f=\infty \quad \text{and} \quad \int^\infty p^{-1}<\infty.$$

Note that by Lemma 3.1, $\lambda(x)<1$ for $a<x<c_1$, $\lambda(x)=1$ for $c_1\leqq x\leqq c_2$ and $0<\lambda(x)<1$ for $c_2<x<\infty$, using the notation of that lemma.

THEOREM 5.1. *If the coefficients of* (1) *satisfy* $\int^\infty f=\infty$ *and* $\int^\infty p^{-1}<\infty$, *in addition to the given conditions, then there exist positive numbers* α *and* β, $0<\alpha\leqq\beta<\infty$, *such that*

(a) $\limsup_{b\to\infty}\lambda(b)\int_a^b f=\beta$ *and, hence* $\lambda(b)\to 0$, *as* $b\to\infty$ *and*

(b) $\liminf_{b\to\infty}\lambda(b)\int_a^b f=\alpha>0$.

PROOF. Let $\theta(x,\lambda)$ be the solution of the system (5b) with $f(x)$ replaced by $\lambda f(x)$ when $0\leqq\lambda<\infty$. Hence, on $a\leqq x<\infty$:

$$\text{(9)} \quad \theta'(x,\lambda)=p^{-1}(x)\cos^2\theta(x,\lambda)+\lambda f(x)\sin^2\theta(x,\lambda),\ \theta(a,\lambda)=0.$$

For $\lambda=0$, $\theta_0(x)=\theta(x,0)$ can be determined explicitly; in fact,

$$\theta_0(x)=\tan^{-1}\left(\int_a^x p^{-1}\right)<\tan^{-1}\left(\int_a^\infty p^{-1}\right)<\pi/2.$$

Let $a<b<\infty$ and $\theta_b(x)=\theta(x,\lambda(b))$. By the methods of [2, Chapter 8] it is established that

$$0 = \theta_b(a) < \theta_b(x) < \theta_b(b) = \pi/2 \quad \text{for} \quad a < x < b.$$

Also, by subtracting (9) with $\lambda=0$ and (9) with $\lambda=\lambda(b)\geqq 0$ it follows that

$$(\theta_b - \theta_0)' + h(x, b)(\theta_b - \theta_0) = \lambda(b) f(x) \sin^2 \theta_b,$$

where

$$h(x, b) = \begin{cases} p^{-1}\left[\dfrac{\sin^2\theta_b - \sin^2\theta_0}{(\theta_b - \theta_0)}\right], \text{ for } \theta_b \neq \theta_0 \\ p^{-1}[\sin 2\theta_0], \text{ for } \theta_b = \theta_0, \end{cases} \quad \text{on } a \leqq x \leqq b$$

and which yields

$$\theta_b(x) - \theta_0(x) = \lambda(b) \int_a^x f(t) \sin^2\theta_b(t) \exp\left(-\int_t^x h\right) dt. \tag{10}$$

Thus, for $a<x<b$:

$$0 = \theta_0(a) = \theta_b(a) < \theta_0(x) < \theta_b(x) < \theta_b(b) = \pi/2.$$

Now, on $a<x<b$, there exists $\bar{\theta}(x)$ such that

$$h(x, b) = p^{-1}(x) \sin 2\bar{\theta}(x) \quad \text{and} \quad 0 < \theta(x) < \bar{\theta}(x) < \theta_b(x) < \pi/2.$$

Hence, $h(x, b)>0$ for $a<x<b$ and $\exp(-\int_t^x h)<1$ for $a\leqq t<x\leqq b$. This inequality in (10) yields

$$0 < \cot^{-1}\left(\int_a^\infty p^{-1}\right) \leqq \cot^{-1}\left(\int_a^b p^{-1}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\int_a^b p^{-1}\right) \leqq \lambda(b) \int_a^b f$$

from which part (b) of Theorem 5.1 follows immediately.

In order to obtain the inequality (a) note that on $a\leqq x<\infty$:

$$|h(x, b)| \leqq p^{-1}(x)$$

and

$$0 < \exp\left(-\int_a^\infty p^{-1}\right) < \exp\left(-\int_a^b p^{-1}\right) \leqq \exp\left(-\int_t^b h\right)$$

which substituted in (10) gives

$$\cot^{-1}\left(\int_a^b p^{-1}\right) = \frac{\pi}{2} - \tan^{-1}\left(\int_a^b p^{-1}\right)$$
$$\geqq \lambda(b) \exp\left(-\int_a^b p^{-1}\right)\int_a^b f(t) \sin^2 \theta_0(t) dt$$
$$\geqq \lambda(b) \exp\left(-\int_a^b p^{-1}\right)\int_a^b \frac{f(t)\left(\int_a^t p^{-1}\right)^2}{1+\left(\int_a^t p^{-1}\right)^2} dt.$$

Let $a<c<b$; then

$$\lambda(b)\int_c^b f$$
$$\leqq \exp\left(\int_a^b p^{-1}\right)\cot^{-1}\left(\int_a^b p^{-1}\right)\left[1+\left(\int_a^c p^{-1}\right)^2\right]\Big/\left(\int_a^c p^{-1}\right)^2.$$

Since $\int^\infty f=\infty$ and $\int^\infty p^{-1}<\infty$, then $\lambda(b)\to 0$ as $b\to\infty$ and (a) follows immediately, completing the proof of Theorem 5.1.

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