

ON THE MATRIX RICCATI EQUATION¹

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1. In this note we give several elementary results concerning the matrix Riccati equation

$$(1.1) \quad \frac{d\Gamma}{dt} + \Gamma G_3 \Gamma + \Gamma G_4 - G_1 \Gamma - G_2 = 0$$

where G_1, G_2, G_3, G_4 are $n_1 \times n_1, n_1 \times n_2, n_2 \times n_1, n_2 \times n_2$ matrices respectively which are continuous on some interval I (which need not be bounded or closed) and where Γ is an $n_1 \times n_2$ matrix. If $n_1 = n_2 = 1$, then (1.1) is the classical (scalar) Riccati equation. The case $n_1 = n_2$ has been considered by Radon [6; 7], Reid [8; 9], Sternberg [11; 12] and Whyburn [15]. The case $n_2 = 1$ has been treated by Chiellini [1] and Coles [2]. Lyaščenko [4; 5] considers the case of arbitrary n_1, n_2 in connection with the problem of splitting a linear system of differential equations into two independent lower-order systems. Applications of (1.1), for $n_1 = n_2$, have been made by Sternberg [13], and by Sternberg and Kaufman [14].

In §2 an explicit formula, (2.2), for the solution of (1.1) in terms of its initial matrix and a fundamental matrix for an associated linear system of differential equations, (2.1), is given. Several theorems relating (1.1) and (2.1) are given in this section. For $n_1 = n_2 = 1$ (2.2) is a well-known result for the scalar Riccati equation. For $n_1 = n_2$ Theorem 1 is equivalent to Theorem 3.1 of [9].

In §3 a theorem (Theorem 5) is obtained for the case $n_1 = n_2 = n$ which reduces when $n = 1$ to the classical cross-ratio property for solutions of the scalar Riccati equation. It is different from a theorem due to Reid [9], which also reduces to the classical cross-ratio property in the scalar case, in that it involves only four solutions of the matrix Riccati equation whereas Reid's theorem involves $n^2 + 3$ solutions. The proof is, however, similar to Reid's as they both follow the classical proof, see Ince [3], and rely heavily on a theorem for linear systems due to Reid [8].

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2. Associated with (1.1) in a basic manner is the $n_1 + n_2$ dimensional linear system

$$(2.1) \quad \frac{d}{dt} x = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} x$$

where the G_i of (2.1) are the same as in (1.1) and where $x = \text{col}(x_1, \dots, x_{n_1+n_2})$. For every $s \in I$ let

$$M(t, s) = \begin{pmatrix} M_1(t, s) & M_2(t, s) \\ M_3(t, s) & M_4(t, s) \end{pmatrix},$$

where M_1, M_2, M_3, M_4 are $n_1 \times n_1, n_1 \times n_2, n_2 \times n_1, n_2 \times n_2$ matrices respectively, be the fundamental matrix for (2.1) such that $M(s, s) = E_{n_1+n_2}$ (the unit $(n_1+n_2) \times (n_1+n_2)$ matrix).

THEOREM 1. *Let $s \in I$ and let $\Gamma(s)$ be a given $n_1 \times n_2$ matrix, then*

$$(2.2) \quad \Gamma(t) = [M_1(t, s)\Gamma(s) + M_2(t, s)][M_3(t, s)\Gamma(s) + M_4(t, s)]^{-1}$$

is, for t sufficiently close to s , the solution of (1.1) having initial matrix $\Gamma(s)$ at $t=s$.

PROOF. The condition $M(s, s) = E_{n_1+n_2}$ is easily seen to guarantee the local existence of the inverse matrix in (2.2). The proof is accomplished by direct substitution of (2.2) into (1.1).

Using any solution matrix of (2.1), not necessarily fundamental, a formula similar to (2.2) can still be constructed. In particular, we have

THEOREM 2. *Let*

$$N(t) = \begin{pmatrix} N_1(t) & N_2(t) \\ N_3(t) & N_4(t) \end{pmatrix}$$

where N_1, N_2, N_3, N_4 are $n_1 \times n_1, n_1 \times n_2, n_2 \times n_1, n_2 \times n_2$ matrices respectively, be a solution matrix for (2.1). If C is a constant $n_1 \times n_2$ matrix, then

$$(2.3) \quad \Gamma(t) = [N_1(t)C + N_2(t)][N_3(t)C + N_4(t)]^{-1}$$

is a solution of (1.1) whenever it exists.

PROOF. Substitute (2.3) into (1.1).

In the scalar case it is well known that, given any four continuously differentiable functions $N_i(t)$ ($i=1, 2, 3, 4$) such that $N_1N_4 - N_2N_3 \neq 0$, (2.3) defines a Riccati equation and is a solution of it for any constant C . The following theorem is a generalization of this result.

THEOREM 3. Let $N_1(t)$, $N_2(t)$, $N_3(t)$, $N_4(t)$ be $n_1 \times n_1$, $n_1 \times n_2$, $n_2 \times n_1$, $n_2 \times n_2$ continuously differentiable matrices on I and let

$$N(t) = \begin{pmatrix} N_1(t) & N_2(t) \\ N_3(t) & N_4(t) \end{pmatrix}$$

be nonsingular on I . Then

$$(2.4) \quad \Gamma(t) = [N_1(t)C + N_2(t)][N_3(t)C + N_4(t)]^{-1}$$

where C is a constant $n_1 \times n_2$ matrix, defines a matrix Riccati equation and is a solution of it whenever (2.4) exists.

PROOF. Define $G(t) = dN(t)/dtN^{-1}(t)$. Then $N(t)$ is a fundamental matrix for $dx/dt = G(t)x$. Partition $G(t)$ so that

$$(2.5) \quad G(t) = \begin{pmatrix} G_1(t) & G_2(t) \\ G_3(t) & G_4(t) \end{pmatrix}$$

where the G_i of (2.5) are of the same size as the G_i of (1.1). Defining a matrix Riccati equation of the form (1.1) with the G_i of (2.5) and applying the preceding theorem completes the proof.

One of the applications of the matrix Riccati equation concerns the local behavior of solutions of nonlinear systems of differential equations. In this connection (2.1) would arise as the first variational equation of a nonlinear system of differential equations with respect to some base solution. Formula (2.2) then plays the following role.

THEOREM 4. Let $s \in I$ and let $\Gamma(s)$ be a given $n_1 \times n_2$ matrix. If

$$\text{col}(x_1(t), \dots, x_{n_1+n_2}(t))$$

is any solution of (2.1) such that

$$\text{col}(x_1(s), \dots, x_{n_1}(s)) = \Gamma(s) \text{col}(x_{n_1+1}(s), \dots, x_{n_1+n_2}(s)),$$

then

$$\text{col}(x_1(t), \dots, x_{n_1}(t)) = \Gamma(t) \text{col}(x_{n_1+1}(t), \dots, x_{n_1+n_2}(t))$$

for t sufficiently close to s , where $\Gamma(t)$ is given by (2.2).

PROOF. It is easily seen that

$$\begin{aligned} \text{col}(x_1(t), \dots, x_{n_1}(t)) &= [M_1(t, s)\Gamma(s) + M_2(t, s)] \text{col}(x_{n_1+1}(s), \dots, x_{n_1+n_2}(s)), \\ \text{col}(x_{n_1+1}(t), \dots, x_{n_1+n_2}(t)) &= [M_3(t, s)\Gamma(s) + M_4(t, s)] \text{col}(x_{n_1+1}(s), \dots, x_{n_1+n_2}(s)) \end{aligned}$$

if $\text{col}(x_1(t), \dots, x_{n_1+n_2}(t))$ is any solution of (2.1) that satisfies the hypothesis. Eliminating $\text{col}(x_{n_1+1}(s), \dots, x_{n_1+n_2}(s))$ from these equations in the obvious manner completes the proof.

3. In this section $n_1 = n_2 = n$. The following result on linear systems, due to Reid [8], is used in the proof of the next theorem.

LEMMA. Let $A(t)$ and $B(t)$ be $n \times n$ continuous matrices for $t \in I$. Let $\Lambda(t)$ and $\Omega(t)$ be $n \times n$ nonsingular matrices that satisfy

$$\frac{d\Lambda}{dt} = A\Lambda, \quad \frac{d\Omega}{dt} = \Omega B \quad (t \in I).$$

Then $P(t)$ satisfies

$$\frac{dP}{dt} = AP + PB \quad (t \in I)$$

if and only if there exists a constant matrix C such that

$$P(t) = \Lambda(t)C\Omega(t) \quad (t \in I).$$

THEOREM 5. If $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ are solutions of (1.1) that are continuous on some common interval $I^* \subset I$, and if there exists a $t_0 \in I^*$ such that $\Gamma_i(t_0) - \Gamma_j(t_0)$ is nonsingular for $i \neq j$, then

$$(3.1) \quad (\Gamma_3 - \Gamma_1)^{-1}(\Gamma_3 - \Gamma_2)(\Gamma_4 - \Gamma_2)^{-1}(\Gamma_4 - \Gamma_1) = \Lambda K \Lambda^{-1} \quad (t \in I^*)$$

where $\Lambda = \Lambda(t)$ and K (a constant matrix) are nonsingular matrices.

PROOF. Define

$$(3.2) \quad U_i = \Gamma_{i+1} - \Gamma_1 \quad (i = 1, 2, 3).$$

Substitution shows that the U_i are solutions of

$$\frac{dU}{dt} + UG_3U + (\Gamma_1G_3 - G_1)U + U(G_3\Gamma_1 + G_4) = 0.$$

Hence, the U_i satisfy $dU/dt = AU + UB$, where $A = -UG_3 - \Gamma_1G_3 + G_1$, $B = -G_3\Gamma_1 - G_4$. Therefore, as $U_i(t_0)$ is nonsingular by hypothesis, it follows from the preceding lemma that $U_i(t)$ is nonsingular for $t \in I^*$. A similar calculation shows that $U_i(t) - U_j(t)$ ($i \neq j$) is nonsingular for $t \in I^*$. Define

$$(3.3) \quad V_i = U_i^{-1} \quad (i = 1, 2, 3).$$

Substitution shows that the V_i are solutions of

$$dV/dt = (G_3\Gamma_1 + G_4)V + V(\Gamma_1G_3 - G_1) + G_3.$$

Define

$$(3.4) \quad W_i = V_{i+1} - V_1 \quad (i = 1, 2).$$

Then the W_i are solutions of

$$dW/dt = (G_3\Gamma_1 + G_4)W + W(\Gamma_1G_3 - G_1).$$

As

$$W_i = V_{i+1} - V_1 = U_{i+1}^{-1} - U_1^{-1} = U_{i+1}^{-1}(U_1 - U_{i+1})U_1^{-1} \quad (i = 1, 2)$$

we see that the W_i are nonsingular for $t \in I^*$. Hence, by the preceding lemma, there exist nonsingular constant matrices C_1 and C_2 such that

$$(3.5) \quad W_i = \Lambda C_i \Omega \quad (t \in I^*), (i = 1, 2)$$

where Λ and Ω are nonsingular matrices that satisfy

$$\frac{d\Lambda}{dt} = (G_3\Gamma_1 + G_4)\Lambda, \quad \frac{d\Omega}{dt} = \Omega(\Gamma_1G_3 - G_1).$$

From (3.2), (3.3), (3.4), and (3.5) one has

$$(3.6) \quad (\Gamma_{i+2} - \Gamma_1)^{-1} - (\Gamma_2 - \Gamma_1)^{-1} = \Lambda C_i \Omega \quad (i = 1, 2).$$

Multiplying (3.6) by $\Gamma_2 - \Gamma_1$ on the right and by $\Gamma_{i+2} - \Gamma_1$ on the left yields

$$\Gamma_2 - \Gamma_{i+2} = (\Gamma_{i+2} - \Gamma_1)\Lambda C_i \Omega(\Gamma_2 - \Gamma_1) \quad (i = 1, 2).$$

Hence, as $\Gamma_2 - \Gamma_3$ and $\Gamma_2 - \Gamma_4$ are nonsingular,

$$\begin{aligned} (\Gamma_2 - \Gamma_3)^{-1}(\Gamma_3 - \Gamma_1)\Lambda C_1 \Omega(\Gamma_2 - \Gamma_1) \\ = (\Gamma_2 - \Gamma_4)^{-1}(\Gamma_4 - \Gamma_1)\Lambda C_2 \Omega(\Gamma_2 - \Gamma_1) \end{aligned}$$

from which (3.1), with $K = C_1 C_2^{-1}$, follows easily.

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