COEFFICIENTS IN CERTAIN ASYMPTOTIC FACTORIAL EXPANSIONS

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1. **Introduction.** Let p and q denote integers with $0 \leqq p \leqq q$, then it is known from [**2**; **3**] and [**7**] that the function

$$g(w) = \prod_{i=1}^{p} \Gamma(w + \sigma_i) \Big/ \prod_{j=0}^{q} \Gamma(w + \rho_j), \tag{1.1}$$

where the σ_i and ρ_j are independent of w, admits for large $|w|$ in the sector $|\arg w| < \pi - \epsilon$, an asymptotic expansion of the form

$$\begin{aligned} g(w) = ((2\pi)^{1/2})^{1-\alpha}\alpha^{\alpha w+\beta-1/2} \Bigg[& \sum_{m=0}^{M} c_m/\Gamma(\alpha w + \beta + m) \\ & + O(1/\Gamma(\alpha w + \beta + M + 1)) \Bigg]. \end{aligned} \tag{1.2}$$

Here, and in the sequel, M denotes a non-negative integer independent of w, ϵ any positive constant and

$$c_0 = 1; \quad \alpha = q + 1 - p; \quad \beta = \sum_{j=0}^{q} \rho_j - \sum_{i=1}^{p} \sigma_i + (1 - \alpha)/2. \tag{1.3}$$

Recently two types of linear recurrence relations satisfied by the c_m have been found. The first [**5**] is an inductive formula for c_m depending on all the preceding c_n; the second [**4**] gives c_m in terms of the preceding q coefficients. An alternative method of obtaining a recursion formula of the latter type has been pointed out by E. M. Wright [**6**].

J. G. van der Corput [**8**] has shown that the inverse of $g(w)$ possesses the asymptotic development

$$\begin{aligned} (g(w))^{-1} = ((2\pi)^{1/2})^{\alpha-1}\alpha^{-\alpha w-\beta+1/2} \Bigg[& \sum_{m=0}^{M} \gamma_m \Gamma(\alpha w + \beta - m) \\ & + O(\Gamma(\alpha w + \beta - M - 1)) \Bigg], \end{aligned} \tag{1.4}$$

in the sector $|\arg w| < \pi - \epsilon$. Moreover, van der Corput has extended

Presented to the Society, August 27, 1958, under the title of *Coefficients in certain asymptotic factorial expansions of the second kind*; received by the editors August 7, 1958 and, in revised form, November 6, 1958.

the method of [5] to obtain a linear recursion formula for γ_m, $\gamma_0=1$, depending on all the preceding γ_n. The purpose here is to present a general method which when applied to $g(w)$ and $(g(w))^{-1}$ yields a finite recursion formula of fixed length q for the c_m and γ_m respectively.[1]

2. **General factorial expansion.** Let us consider the function

$$F(x) = \prod_{j=0}^{q} \Gamma(x+s_j)/\Gamma(x+r_j), \tag{2.1}$$

where the s_j and r_j are independent of x. From Stirling's formula it is easy to show that there exist constants A_m, $A_0=1$, such that in the sector $|\arg x| \leqq \pi-\epsilon F(x) = \sum_{m=0}^{M} A_m x^{-b-m} + O(x^{-b-M-1})$, where

$$b = \sum_{j=0}^{q} (r_j - s_j). \tag{2.2}$$

The proof which is similar to the proofs cited for (1.2) is omitted. We immediately have the following result which will be stated in the form of a theorem.

THEOREM 2.1. *Let d_m and a be given complex numbers, $|\arg a| < \pi/2$. Then there exist constants B_m, $B_0=1$, such that*

$$F(x/a) = a^b \sum_{m=0}^{M} B_m \Gamma(x+d_m-m)/\Gamma(x+d_m+b) + O(x^{-b-M-1}) \tag{2.3}$$

in the sector $|\arg x| < \pi/2+\epsilon$ provided $\epsilon < \pi/2 - |\arg a|$.

The problem is to determine an efficient method for computing the coefficients B_m. For this purpose we shall need the general result of the next section.

3. **A general theorem.** Let $G(x)$ denote a function analytic in some sector $|\arg (x-x_0)| \leqq \pi/2+\epsilon$, $(\epsilon<\pi/2)$, such that in this sector

$$G(x) = O(x^{-h}), \tag{3.1}$$

where h denotes a constant. For such a function, we define

$$\psi(t) = L\{G\} = (2\pi i)^{-1} \int_C t^{\lambda x} G(x)\,dx, \qquad \begin{array}{l}(1 \leqq t < \infty \text{ if } \lambda > 0),\\ (0 < t \leqq 1 \text{ if } \lambda < 0),\end{array} \tag{3.2}$$

where λ $(\neq 0)$ is real and C denotes the path $|\arg (x-x_0)| = \pi/2+\epsilon$ orientated in the upward direction; clearly, $\psi(t)$ does not depend on

[1] We had originally demonstrated the result for γ_m. This method of giving the general result was suggested by the referee.

the choice of ϵ or x_0. Further, $\psi(t)$ is infinitely differentiable, in fact

$$(3.3)\quad \psi^{(k)}(t) = \left(\frac{d}{dt}\right)^k L\{G\} = t^{-k}L\{\lambda x(\lambda x - 1)\cdots(\lambda x - k + 1)G(x)\}.$$

Finally, we note the following result:

THEOREM 3.1. *$G(x) = O(x^{-h})$ implies that near $t = 1$*

$$(3.4)\qquad \psi^{(k)}(t) = O(|t - 1|^{h-k-1}),$$

provided $k < \mathrm{Re}\,(h) - 1$.

PROOF. In the integral representation of $\psi^{(k)}(t)$ replace the curve C by $\mathrm{Re}\,(x) = u \geqq u_0 = \mathrm{Re}\,(x_0)$ and set $x = u + iv$ to obtain

$$\psi^{(k)}(t) = (2\pi)^{-1}t^{\lambda u-k}\int_{-\infty}^{+\infty} t^{i\lambda v}(\lambda u + i\lambda v)(\lambda u - 1 + i\lambda v)\cdots$$
$$(\lambda u - k + 1 + i\lambda v)G(u + iv)dv.$$

There exists a constant K, independent of u and v, such that

$$|\psi^{(k)}(t)| < K\int_{-\infty}^{+\infty} t^{\lambda u-k}((u^2 + v^2)^{1/2})^{k-\mathrm{Re}\,h}\,dv.$$

Substitute $v^2 = wu^2$ to write the last relation in the form

$$|\psi^{(k)}(t)| < (K/2)u^{k-\mathrm{Re}\,h+1}\int_{-\infty}^{+\infty} t^{\lambda u-k}((1 + w)^{1/2})^{k-\mathrm{Re}\,h}w^{-1/2}dw.$$

Since the integral is convergent, the assertion follows immediately upon setting $u = |t-1|^{-1}$.

4. **The function $\phi(t)$.** First apply Theorem 3.1 to the order term in (2.3) to find that, near $t = 1$,

$$\left(\frac{d}{dt}\right)^h L\left\{F(x/a) - a^b\sum_{m=0}^{M} B_m\Gamma(x + d_m - m)/\Gamma(x + d_m + b)\right\}$$
$$= O(|t - 1|^{b+M-k}),$$

provided $k < M + \mathrm{Re}\,(b)$. It now follows upon introducing

$$(4.1)\qquad \phi(t) = L\{F(x/a)\},\qquad \begin{array}{l}(1 \leqq t < \infty \text{ if } \lambda > 0),\\ (0 < t \leqq 1 \text{ if } \lambda < 0),\end{array}$$

and using the well-known relation [1, p. 261]

$$L\{\Gamma(x + d_m - m)/\Gamma(x + d_m + b)\}$$
$$= (\Gamma(m + b))^{-1}t^{-\lambda(d_m+b-1)}(t^\lambda - 1)^{m+b-1},$$

that

$$\phi^{(k)}(t) = a^b \sum_{m=0}^{M} B_m(\Gamma(m+b))^{-1}\left(\frac{d}{dt}\right)^k [t^{-\lambda(d_m+b-1)}(t^\lambda - 1)^{m+b-1}] + O(|t-1|^{b+M-k}), \tag{4.2}$$

provided $k < M + \mathrm{Re}\,(b)$.

Now note that by applying the recurrence relation $\Gamma(x+1) = x\Gamma(x)$ to (2.1), we obtain

$$F(x/a+1)\prod_{j=0}^{q}(x+ar_j) = F(x/a)\prod_{j=0}^{q}(x+as_j). \tag{4.3}$$

There exist constants C_k and E_k $(k=0, 1, \cdots, q+1)$ such that

$$\begin{aligned} \prod_{j=0}^{q}(x+as_j) &= \sum_{k=0}^{q+1} C_k\lambda x(\lambda x-1)\cdots(\lambda x-k+1), \\ \prod_{j=0}^{q}(x+ar_j) &= \sum_{k=0}^{q+1} E_k(\lambda x+\lambda a)(\lambda x+\lambda a-1)\cdots \\ &\qquad (\lambda x+\lambda a-k+1). \end{aligned} \tag{4.4}$$

Relations (4.3) and (4.4) are used to prove the assertion which follows.

THEOREM 4.1. *The function $\phi(t)$ defined by* (4.1) *satisfies the linear differential equation*

$$\sum_{k=0}^{q+1}(t^{k+\lambda a}C_k - t^k E_k)\phi^{(k)}(t) = 0, \tag{4.5}$$

where

$$\begin{aligned} C_k &= (k!)^{-1}\prod_{j=0}^{q}(as_j + k/\lambda) - \sum_{n=0}^{k-1} C_n/(k-n)!, \\ E_k &= (k!)^{-1}\prod_{j=0}^{q}(ar_j - a + k/\lambda) - \sum_{n=0}^{k-1} E_n/(k-n)!. \end{aligned} \tag{4.6}$$

In particular

$$\begin{aligned} C_{q+1} &= \lambda^{-q-1}; \quad C_q = \lambda^{-q}\left[a\sum_{j=0}^{q} s_j + q(q+1)(2\lambda)^{-1}\right], \\ E_{q+1} &= \lambda^{-q-1}; \quad E_q = \lambda^{-q}\left[a\sum_{j=0}^{q} r_j + (q-2a\lambda)(q+1)(2\lambda)^{-1}\right]. \end{aligned} \tag{4.7}$$

PROOF. From (3.3) and (4.1) we may write

$$\begin{aligned}\sum_{k=0}^{q+1} E_k t^k \phi^{(k)}(t) &= L\left\{F(x/a) \sum_{k=0}^{q+1} E_k \lambda x(\lambda x - 1) \cdots (\lambda x - k + 1)\right\} \\ &= t^{\lambda a} L\left\{F(x/a + 1) \sum_{k=0}^{q+1} E_k(\lambda x + \lambda a)(\lambda x + \lambda a - 1)\right. \\ &\qquad \left. \cdots (\lambda x + \lambda a - k + 1)\right\} \\ &= t^{\lambda a} L\left\{F(x/a) \sum_{k=0}^{q+1} C_k \lambda x(\lambda x - 1) \cdots (\lambda x - k + 1)\right\} \\ &= t^{\lambda a} \sum_{k=0}^{q+1} C_k t^k \phi^{(k)}(t),\end{aligned}$$

which proves assertion (4.5). Equations (4.6) are obtained immediately upon substituting the appropriate values of x into the identities (4.4). Relations (4.7) are obtained by comparing the coefficients of x^{q+1} and x^q in the identities (4.4).

5. **Recursion formulae.** A general technique for obtaining the constants B_m in (2.3) is to substitute the asymptotic expansions (4.2) in (4.5), expand all the resulting terms about $t=1$, and equate the coefficients of like powers. This procedure will first be applied to find a finite recursion formula for the constants γ_m which occur in (1.4).

The method requires that we write (1.4) in the form of (2.3) with $b>q+1$. We choose $b=q+3$ to be consistent with the notation of [4]. In (1.4) set $x=\alpha w+\beta+q+2$ and use the identity

$$((2\pi)^{1/2})^{\alpha-1}\Gamma(x + 1) = \alpha^{x+1/2} \prod_{\mu=1}^{\alpha} \Gamma(x/2 + \mu/2)$$

to find

$$(5.1)\quad F_1(x/\alpha) = \alpha^{q+3} \sum_{m=0}^{M} \gamma_m \Gamma(x - q - 2 - m)/\Gamma(x + 1) + O(x^{-q-4-M}),$$

where

$$\begin{aligned}F_1(x) = &\prod_{j=0}^{q} \Gamma(x + \rho_j - \alpha^{-1}(\beta + q + 2)) \\ &\cdot \left(\prod_{i=1}^{p} \Gamma(x + \sigma_i - \alpha^{-1}(\beta + q + 2)) \prod_{\mu=1}^{\alpha} \Gamma(x + \alpha^{-1}\mu)\right)^{-1}.\end{aligned}$$

Equation (5.1) is in the form of (2.3) with

$$(5.2)\qquad a = \alpha; \qquad b = q + 3; \qquad d_m = -q - 2 (m = 0, 1, \cdots),$$

$$(5.3)\qquad \begin{aligned} s_j &= \rho_j - \alpha^{-1}(\beta + q + 2) && (j = 0, 1, \cdots q), \\ r_j &= \sigma_{j+1} - \alpha^{-1}(\beta + q + 2) && (j = 0, 1, \cdots, p - 1), \\ &= \alpha^{-1}(j - p + 1) && (j = p, \cdots, q). \end{aligned}$$

If now we choose $\lambda = 1$ in (4.2) we have simply

$$(5.4)\qquad \phi^{(k)}(t) \sim \alpha^{q+3} \sum_{m=0}^{\infty} \frac{\gamma_m (t - 1)^{m+q+2-k}}{\Gamma(m + q + 3 - k)} \qquad (1 \leqq t < \infty).$$

Substitution of (5.4) into (4.5) yields the desired recurrence relation for the constants γ_m.

THEOREM 5.1. *The coefficients* γ_m $(m = 0, 1, \cdots)$, *occurring in the expansion* (1.4), *satisfy the following recursion formula of length* q:

$$(5.5)\qquad \begin{aligned} \gamma_m = {} & (\alpha m)^{-1}(m + 2)! \\ & \cdot \left\{ \sum_{i=1}^{q} \sum_{k=q-i}^{q+1} C_k \binom{k}{q - i} \gamma_{m-i}/(m + q - i + 2 - k)! \right. \\ & \left. - \sum_{i=1}^{p-1} \sum_{k=q-i}^{q+1} E_k \binom{k - \alpha}{q - i - \alpha} \gamma_{m-i}/(m + q - i + 2 - k)! \right\}, \end{aligned}$$

($\gamma_0 = 1$ *and* $\gamma_{-1} = \gamma_{-2} = \cdots = 0$), *where*

$$(5.6)\qquad \begin{aligned} C_k &= (k!)^{-1} \prod_{j=0}^{q} (\alpha\rho_j - \beta - q - 2 + k) - \sum_{n=0}^{k-1} C_n/(k - n)!, \\ E_k &= (k!)^{-1} \prod_{j=0}^{p-1} (\alpha\sigma_{j+1} - \beta - q - 2 - \alpha + k) \prod_{j=p}^{q} (j - q + k) \\ &\quad - \sum_{n=0}^{k-1} E_n/(k - n)!. \end{aligned}$$

In particular,

$$(5.7)\qquad \begin{aligned} C_{q+1} &= 1; \qquad C_q = \alpha \sum_{j=0}^{q} \rho_j - (q + 1)(\beta + q + 2) + q(q + 1)/2, \\ E_{q+1} &= 1; \qquad E_q = \alpha \sum_{i=1}^{p} \sigma_i - p(\alpha + \beta + q + 2) \\ &\qquad + q(q + 1)/2 - \alpha(\alpha - 1)/2. \end{aligned}$$

PROOF. Equations (5.6) and (5.7) are immediately obtained by the substitution of (5.2) and (5.3) into (4.6) and (4.7). To find the recursion formula (5.5) set $\lambda = 1$ and employ the binomial theorem to write (4.5) in the form

$$\sum_{j=0}^{q+1}\sum_{k=j}^{q+1} C_k \binom{k}{j} (t-1)^{k-j}\phi^{(k)}(t) - \sum_{j=\alpha}^{q+1}\sum_{k=j}^{q+1} E_k \binom{k-\alpha}{j-\alpha} (t-1)^{k-j}\phi^{(k)}(t) = 0.$$

By (5.4) the coefficient of t^{m+2} in the asymptotic expansion for $(t-1)^{k-j}\phi^{(k)}(t)$ is

$$\alpha^{q+3}\gamma_{m-q+j}((m+j+2-k)!)^{-1},$$

so that, for each integer m,

$$\sum_{j=0}^{q+1}\sum_{k=j}^{q+1} C_k \binom{k}{j} \gamma_{m-q+j}((m+j+2-k)!)^{-1} - \sum_{j=\alpha}^{q+1}\sum_{k=j}^{q+1} E_k \binom{k-\alpha}{j-\alpha} \gamma_{m-q+j}((m+j+2-k)!)^{-1} = 0,$$

provided we agree that $\gamma_m = 0$ for $m < 0$. From (5.7), the coefficients of γ_{m+1} and γ_m are 0 and $\alpha m/(m+2)!$ respectively. The above formula therefore reduces to (5.5) upon solving for γ_m and setting $i = q - j$.

The general method may also be applied to obtain a recursion formula for the constants c_m which appear in (1.2). In this case set $x = \alpha w + \beta - q - 3$ and write (1.2) as

$$F_2(x/\alpha) = \alpha^{q+3} \sum_{m=0}^{M} c_m \Gamma(x)/\Gamma(x+q+3+m) + O(x^{-q-4-M}), \tag{5.8}$$

where

$$F_2(x) = \prod_{i=1}^{p} \Gamma(x + \sigma_i - \alpha^{-1}(\beta - q - 3)) \cdot \prod_{\mu=1}^{\alpha} \Gamma(x + \alpha^{-1}(\mu - 1)) \Big/ \prod_{j=0}^{q} \Gamma(x + \rho_j - \alpha^{-1}(\beta - q - 3)).$$

Now (5.8) is in the form of (2.3) with

$$a = \alpha; \qquad b = q + 3; \qquad d_m = m, \tag{5.9}$$

$$\begin{aligned} s_j &= \sigma_{j+1} - \alpha^{-1}(\beta - q - 3) && (j = 0, 1, \cdots, p-1) \\ &= \alpha^{-1}(j - p) && (j = p, \cdots, q), \\ r_j &= \rho_j - \alpha^{-1}(\beta - q - 3) && (j = 0, 1, \cdots, q). \end{aligned} \tag{5.10}$$

If we choose $\lambda = -1$, then (4.2) and (4.5) become simply

$$\phi^{(k)}(t) \sim \alpha^{q+3} \sum_{m=0}^{\infty} (-1)^{m+q} c_m (t-1)^{m+q+2-k} / \Gamma(m+q+3-k) \qquad (0 < t \leqq 1) \tag{5.11}$$

and

$$\sum_{k=0}^{q+1} (t^{k-\alpha} C_k - t^k E_k) \phi^{(k)}(t) = 0 \tag{5.12}$$

respectively. The recursion formula for the c_m obtained by substituting (5.11) into (5.12) is precisely the result given in [**4**].

6. **Remark.** When Theorem (5.1) is applied to the special case of

$$(g(w))^{-1} = \Gamma(w + \rho_0)\Gamma(w + \rho_1) \tag{6.1}$$

the result is

$$\gamma_m (8m)^{-1} [4(\rho_1 - \rho_0)^2 - (2m-1)^2] \gamma_{m-1}. \tag{6.2}$$

On the other hand, van der Corput's formula [**8**, Equation 8] gives

$$\gamma_1 = (8)^{-1} [4(\rho_1 - \rho_0)^2 + 71] \gamma_0$$

for the second coefficient. This discrepancy has been resolved by H. O. Pollak who has computed γ_1 for this special case by substituting Stirling's formula into (6.1) directly. The result is in agreement with (6.2).

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