ON THE NOTION OF ANALYTICITY

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Let E_n be a normed n-dimensional vector space over a field Γ. We denote by $a, b, \cdots$ the elements of E_n, with $\alpha, \beta, \cdots$ the elements of Γ; $a=\alpha_1 e_1+\cdots+\alpha_n e_n \sim \{\alpha_1, \cdots, \alpha_n\}$ where $e_1, \cdots, e_n$ is a basis of E_n. The norm in E_n we denote by $|\ |_n$.

By $a=L\circ b$ we denote a linear homogeneous transformation on E_n to E_n, i.e. $a\in E_n$ is defined for every $b\in E_n$; $L\circ(\lambda b)=\lambda L\circ b$; $L\circ(b_1+b_2)=L\circ b_1+L\circ b_2$ for every λ, b, b_1, b_2. Naturally, given a basis $e_1, \cdots, e_n$ of E_n, L can be represented as a square matrix: $L\sim[\lambda_{\mu\nu}]$ such that $\alpha_\mu=\lambda_{\mu 1}\beta_1+\cdots+\lambda_{\mu n}\beta_n$ $(\mu=1, \cdots, n)$ for $a=L\circ b$, $a\sim\{\alpha_1, \cdots, \alpha\}$, $b=\{\beta_1, \cdots, \beta_n\}$.

Let $y=f(x)$ be a function on E_n to E_n, i.e. $y\in E_n$ for some $x\in E_n$. $f(x)$ is called *differentiable* (in the sense of *Fréchet*) at $x=x_0$ if for $h\in E_n$, $|h|_n\to 0$ there is a $L=L(x_0)$ such that

$$(1)\quad f(x_0+h)=f(x_0)+L(x_0)\circ h+r(h);\ r(h)\in E_n,\ |r(h)|_n=o(|h|_n).$$

If, with respect to the basis $e_1, \cdots, e_n$ it is $f\sim\{\phi_1, \cdots, \phi_n\}$, $x\sim\{\xi_1, \cdots, \xi_n\}$, $L\sim[\lambda_{\mu\nu}]$ then obviously

$$(2)\qquad \lambda_{\mu\nu}=\partial\phi_\mu/\partial\xi_\nu \qquad (x=x_0).$$

We call $\otimes$ *a product on* E_n if to every $a\in E_n$, $b\in E_n$ there exists $c=a\otimes b\in E_n$ such that $a\otimes b=L_a\circ b=L_b\circ a$. For some fixed basis $e_1, \cdots, e_n$ with $a\sim\{\alpha_1, \cdots, \alpha_n\}$, $b\sim\{\beta_1, \cdots, \beta_n\}$, $c\sim\{\gamma_1, \cdots, \gamma_n\}$ such a product can always be realized by n^3 constants $\pi_{\rho\mu\nu}$ such that

$$(3)\qquad \gamma_\rho=\sum_{\mu,\nu=1}^{n}\pi_{\rho\mu\nu}\alpha_\mu\beta_\nu \qquad (\rho=1, \cdots, n).$$

Naturally, by introducing a product in a vector space this vector space becomes an algebra.

If $f(x)$ is a function on E_n to E_n we call it *analytic with respect to the product* $\otimes$ *at* $x=x_0$ if it is there differentiable and if there is an $l=l(x_0)\in E_n$ such that

$$(4)\qquad L(x_0)\circ h=l(x_0)\otimes h,$$

Received by the editors December 21, 1957 and, in revised form, November 10, 1958.

where $L(x_0)$ is defined by (1). Without referring to the definition of differentiability this means: there is an $l(x_0)$ such that

$$(5)\qquad \begin{aligned} f(x_0+h) &= f(x_0) + l(x_0)\otimes h + r(h), && r(h)\in E_n,\\ |r(h)|_n &= o(|h|_n) && \text{for } |h|_n\to 0. \end{aligned}$$

It is obvious that this definition of analyticity contains the one for functions of a complex variable.

Let $l(x_0)\sim\{\lambda_1(x_0),\cdots,\lambda_n(x_0)\}$, $h\sim\{\eta_1,\cdots,\eta_n\}$ for a basis $e_1,\cdots,e_n$. Putting $l(x_0)\otimes h=k$, $k\sim\{\kappa_1,\cdots,\kappa_n\}$ we have $\kappa_\rho = \sum_{\mu,\nu=1}^n \pi_{\rho\mu\nu}\lambda_\mu(x_0)\eta_\nu$ $(\rho=1,\cdots,n)$. On the other hand, the ρth component of $L(x_0)\circ h$ is given by

$$\lambda_{\rho 1}\eta_1+\cdots+\lambda_{\rho n}\eta_n = \partial\phi_\rho/\partial\xi_1\cdot\eta_1+\cdots+\partial\phi_\rho/\partial\xi_n\cdot\eta_n.$$

Instead of (4) we have therefore

$$(6)\qquad \partial\phi_\rho/\partial\xi_\nu = \pi\rho_{1\nu}\lambda_1(x_0)+\cdots+\pi_{\rho n\nu}\lambda_n(x_0)\qquad(\rho,\nu=1,\cdots,n).$$

We put

$$\Pi_\nu = \operatorname{Det}\pi_{\rho\mu\nu} = \begin{vmatrix} \pi_{11\nu} & & \\ & \ddots & \\ & & \pi_{nn\nu}\end{vmatrix}\qquad(\nu=1,\cdots,n)$$

and denote by $\Pi_{\nu\rho}$ the determinant we get from Π_ν by replacing its ρth column by $\partial\phi_1/\partial\xi_\nu,\cdots,\partial\phi_n/\partial\xi_\nu$. It follows

$$(7)\qquad \Pi_{1\rho}/\Pi_1 = \cdots = \Pi_{n\rho}/\Pi_n(=\lambda_\rho)\qquad(\rho=1,\cdots,n),$$

provided $\Pi_\nu\neq 0$ $(\nu=1,\cdots,n)$. If some of these determinants vanish, $\Pi_1=\cdots=\Pi_k=0$, $k\leqq n$ say, then (7) has to be replaced by

$$(8)\qquad \begin{aligned} \Pi_{1\rho} &= \cdots = \Pi_{k\rho} = 0,\\ \Pi_{k+1,\rho}/\Pi_{k+1} &= \cdots = \Pi_{n\rho}/\Pi_n \end{aligned}\qquad(\rho=1,\cdots,n).$$

The equations (7) viz (8) form a necessary condition for the analyticity of $f(x)$ at $x=x_0$. We are therefore entitled to consider them as a generalization of the Cauchy-Riemann differential equations.

In the case $n=2$ and complex multiplication, the equations (7) yield the Cauchy-Riemann equations. In the case $n=4$, E_n being a real space and the product being defined according to the multiplication of quaternions, these equations read

$$\phi_{11} = \phi_{22} = \phi_{33} = \phi_{44},$$
$$\phi_{21} = -\phi_{12} = \phi_{43} = -\phi_{34},$$
$$\phi_{31} = -\phi_{42} = -\phi_{13} = \phi_{24},$$
$$\phi_{41} = \phi_{32} = -\phi_{23} = -\phi_{14}$$

where we have abbreviated $\partial\phi_\nu/\partial\xi_\mu$ by $\phi_{\nu\mu}$. In the case $n=3$, E_n being real and the product being the ordinary vector product, the function $f(\xi_1, \xi_2, \xi_3) = \{\xi_2-\xi_3, \xi_3-\xi_1, \xi_1-\xi_2\}$ turns out to be everywhere analytic.

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