# DIFFEOMORPHISMS OF THE 2-SPHERE

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The object of this paper is to prove the theorem.

THEOREM A. *The space* $\Omega$ *of all orientation preserving* $C^\infty$ *diffeomorphisms of* $S^2$ *has as a strong deformation retract the rotation group* $SO(3)$.

Here $S^2$ is the unit sphere in Euclidean 3-space, the topology on $\Omega$ is the $C^r$ topology $\infty \geqq r > 1$ (see [4]) and a diffeomorphism is a differentiable homeomorphism with differentiable inverse.

The method of proof uses

THEOREM B. *The space* $\mathfrak{F}$ ($C^r$ *topology*) *of* $C^\infty$ *diffeomorphisms of the unit square which are the identity in some neighborhood of the boundary is contractible to a point.*

The analogue of Theorem A for the topological case was proved by H. Kneser [2]. The problem in his case seems to be of a different nature from the differentiable case. J. Munkres [3] has proved that $\Omega$ is arcwise connected.

Conversations with R. Palais have been helpful in the preparation of this paper.

Let $I^2$ be the square in the Euclidean plane $E^2$ with coordinate $(t, x)$ such that $(t, x) \in I^2$ if $0 \leqq t \leqq 1$ and $0 \leqq x \leqq 1$. Let $\bar{e}: I^2 \to I^2$ denote the identity diffeomorphism and $\mathfrak{F}$, the space of diffeomorphisms, with the $C^r$ topology, of $I^2$ onto $I^2$ which agree with $\bar{e}$ on some neighborhood of $\dot{I}^2$, the boundary of $I^2$. The $C^r$ topology is such that two maps are close with respect to it if they are close and their first $r$ derivatives are close. See R. Thom [4] for details. We assume that $r$ is fixed in this paper, $\infty \geqq r > 1$, and that all function spaces considered possess the $C^r$ topology. We further assume that all diffeomorphisms are $C^\infty$.

Let $I_1 \subset I^2$ denote the subset $\{(t, x) \in I^2 \mid t = 1\}$, $df_p$ be the differential of a diffeomorphism $f$ at $p$, and $u_0$ be the vector $(1, 0)$ in $E^2$ considered as its own tangent vector space. Then denote by $\mathcal{E}$ the space of diffeomorphisms of $I^2$ onto $I^2$ such that if $f \in \mathcal{E}$, then (a) $f = \bar{e}$ on some neighborhood of $\dot{I}^2 - I_1$, and (b) $df_p(u_0) = u_0$ for all $p$ in some neighborhood of $I_1$.

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Let $\bar{e}_0$: $I^2 \to u_0$ be the constant map and define $\mathcal{S}$ to be the space with the compact open topology of maps of $I^2$ into $S$ which agree with $\bar{e}_0$ in some neighborhood of $\dot{I}^2$, where $S = E^2 - (0, 0)$.

A map $\phi$: $\mathcal{E} \to \mathcal{S}$ is defined as follows:

$$\phi(f)(t, x) = df_{f^{-1}(t,x)}(u_0), \qquad f \in \mathcal{E}.$$

LEMMA 1. *There is a homotopy* $\phi_v$: $\mathcal{E} \to \mathcal{S}$ *such that for each* $f \in \mathcal{E}$,

(a) $\phi_v(f)(t, x)$ *is* $C^\infty$ *in* $(v, x, t)$,

(b) $\phi_0(f) = \bar{e}_0$,

(c) $\phi_1 = \phi$, *and*

(d) $\phi_v(\bar{e}) = \bar{e}_0$.

PROOF. Let $p$: $R \to S$ be the covering map where $R$ is the universal covering space of $S$, and let $\bar{u} \in p^{-1}(u_0)$. Let $T_v$: $R \to R$ be a differentiable contraction of $R$ to $\bar{u}$.

Define now a homotopy $h_v$: $\mathcal{S} \to \mathcal{S}$ by $h_v(f)(x) = pT_v(\bar{f}(x))$ where $\bar{f}$ is the unique lifting of $f$ taking $\dot{I}^2$ into $\bar{u}$. Then it is easily checked that $\phi_v = h_v \circ \phi$ may be taken as our desired homtopy.

LEMMA 2. *There is a homotopy* $H_v$: $\mathcal{E} \to \mathcal{E}$ *such that for each* $f \in \mathcal{E}$,

(a) $H_v(f)$ *is* $C^\infty$ *in* $(v, x, t)$,

(b) $H_0(f) = \bar{e}$,

(c) $H_1(f) = f$, *and*

(d) $H_v(\bar{e}) = \bar{e}$.

PROOF. Let $\phi_v(f)(t, x)$ be considered as a vector field on $I^2$ as given by Lemma 1, for each $f \in \mathcal{E}$ and $v \in I$, $I$ the unit interval. Let $P_v(f)(u, t_0, x_0)$ be the integral curve of $\phi_v(f)$ with the initial condition $P_v(f)(0, t_0, x_0) = (t_0, x_0)$.

Define $Q_v(f)(t, x) = P_v(f)(t, 0, x)$.

Now suppose there were an integral curve $P_v(f)(u, t_0, x_0)$ starting at $(t_0, x_0) = (0, x)$ which did not leave $I^2$. If this were so then it would approach asymptotically some simple closed curve in $I^2$. In the interior of this curve the vector field $\phi_v(f)$ would have to have a singularity. But this is impossible. Thus we conclude that there is a $t$, say $\bar{t}$, with $Q_v(f)(\bar{t}, x)$ meeting $I_1$. This is exactly the part of the proof of Theorem A which does not extend to the case of $S^n$.

Denote the above $\bar{t}$ by $\bar{t}(v, f, x)$. Then $\bar{t}(v, f, x)$ is $C^\infty$ in $(v, x)$ continuous $(v, f, x)$ and positive.

We need the following lemma which can be found for example in [1, p. 172].

LEMMA 3. *Let $g$ be a real lower semi-continuous positive function on a paracompact space $X$. Then there is a real continuous function $h$ on $X$ such that for all* $x \in X$, $0 < h(x) < g(x)$.

Let $g$ be the function on $\mathcal{E}$ defined by

$$g(f) = \min \left\{ \frac{\bar{t}(v, f, x)}{1 - \bar{t}(v, f, x)}, 1 \,\middle|\, v, x \in I, \bar{t}(v, f, x) < 1 \right\}.$$

Then let $\eta$ be the function on $\mathcal{E}$ given by Lemma 3.

Let $\gamma$ be a real function on $\mathcal{E} \times R$, $C^\infty$ in $t$ such that $\gamma(f, t) = 0$ for $t$ in some neighborhood of 0, $\gamma(f, t) = 1$ for $t$ in some neighborhood of 1 and

$$0 < \frac{d\gamma(f, t)}{dt} < 1 + \eta(f).$$

We leave to the reader the task of showing that such a function exists.

Now define $H_v: \mathcal{E} \to \mathcal{E}$ by

$$H_v(f)(t, x) = Q_v(f)(t + \gamma(f, t)(\bar{t}(v, f, x) - 1), x).$$

We prove now that $H_v(f): I^2 \to I^2$ is regular (has Jacobian of rank 2). Note that $H_v(f)$ can be written as the composition $\psi \circ g$ where

$$g: (t, x) \to (t + \gamma(f, t)(\bar{t}(v, f, x) - 1), x) = (t', x'),$$
$$\psi: (t', x') \to Q_v(f)(t', x').$$

From the choice of $\eta$ it follows that $\partial t'/\partial t \neq 0$, and hence $g$ is regular.

Now we prove that $\psi$ is regular. Let

$$\varphi(u, t, x) = P_v(f)(u, t, x) \quad \text{and} \quad \varphi^i(u, t, x), \qquad i = 1, 2$$

be the coordinates of $\varphi$. Also let $X(t, x)$ denote the vector field $\phi_v(f)$ with

$$X(t, x) = X_1(t, x) \frac{\partial}{\partial t} + X_2(t, x) \frac{\partial}{\partial x}.$$

Then $\psi(t, x) = \varphi(t, 0, x)$. It is sufficient to prove $\psi^{-1}$ is differentiable. Let $\tau(t, x)$ be the unique $u$ such that $\varphi^1(u, t, x) = 0$. Then $\psi^{-1}(t, x) = (-\tau(t, x), \varphi^1(\tau(t, x), t, x), \varphi^2(\tau(t, x), t, x))$. The map $\psi^{-1}$ is differentiable if $\tau$ is and $\tau$ is differentiable if

$$\left. \frac{\partial \varphi^1(u, t, x)}{\partial u} \right|_{u = \tau(t, x)} \neq 0.$$

But

$$\left. \frac{\partial \varphi^1(u, t, x)}{\partial u} \right|_{u = \tau(t, x)} = X_1(\varphi(\tau(t, x), t, x) = X_1(0, \varphi^2) = 1.$$

It is easy now to check that $H_v$ is our required homotopy.

The following amplifies Theorem B.

THEOREM 4. *The space $\mathfrak{F}$ is contractible. In fact there is a homotopy $G_v\colon \mathfrak{F}\to\mathfrak{F}$ such that for each $f\in\mathfrak{F}$,*

(a) *$G_v(f)(x, t)$ is $C^\infty$ in $(v, x, t)$,*

(b) *$G_0(f)=\bar{e}$,*

(c) *$G_1(f)=f$, and*

(d) *$G_v(\bar{e})=\bar{e}$.*

PROOF. Let $\mathfrak{F}_1$ be the space of diffeomorphisms of $I$ into $I$ which agree with the identity in some neighborhood of the boundary $\dot{I}$. Let $K_v\colon \mathfrak{F}_1\to\mathfrak{F}_1$ be defined by $K_v(f)(t)=f(t)v+t(1-v)$. Then $K_v(f)(t)$ is $C^\infty$ in $(t, v)$, $K_0(f)(t)=t$, $K_1(f)=f$, and if $e$ is the identity $K_v(e)=e$.

Let $H_v$ be as in Lemma 2, and define $h_v=H_v(h)$ for each $h\in\mathcal{E}$. Let $\bar{h}_v$ denote $h_v$ restricted to $I_1$. Let $\beta(t)$ be a $C^\infty$ function of $t$ such that $\beta(t)=0$ in a neighborhood of 0, $\beta(t)=0$ in a neighborhood of 1.

The desired homotopy $G_v\colon \mathfrak{F}\to\mathfrak{F}$ is defined as follows.

$$G_v(h)(t, x) = (t', [K_{\beta(t')}(\bar{h}_v^{-1})](x'))$$

where $t'$ and $x'$ are the $t$ and $x$ components respectively of $h_v(t, x)$. The map $(t, x)\to(t', x')$ is a diffeomorphism because $h_v$ is. It is easily checked that

$$(t', x') \to (t', [K_{\beta(t')}(\bar{h}_v^{-1})](x'))$$

is a diffeomorphism. Hence the composition $G_v(h)$ is also a diffeomorphism. One can further check that $G_v(h)$ satisfies all the properties demanded by the theorem.

Let $S^2$ be the unit sphere in $E^3$ with Cartesian coordinates $(x, y, z)$ and $x_0$ the South Pole. Let $e_1$, $e_2$ be unit tangent vectors of $S^2$ at $x_0$ in the directions of the $x$ and $y$ axes respectively. Let $\Omega_0$ be the space of diffeomorphisms of $S^2$ such that if $f\in\Omega_0$ then $f(x_0)=x_0$ and $df_{x_0}(e_i)=e_i$, $i=1, 2$.

THEOREM 5. *The space $\Omega_0$ is contractible in the sense of Theorem* 4.

PROOF. Let $E$ be the open southern hemisphere of $S^2$. Then $E^3$ induces a natural Euclidean coordinate system on $E$ and the tangent bundle $T$ of $E$. We will use these coordinates to add points on $E$ and $T$.

Let $g\colon \Omega_0\to R$ be defined by

$$g(f) = \sup \{q \leqq 1 \mid \text{ for all } X \in T(N_q(x_0)),\ \lvert df(X) - X\rvert < 1\}$$

where $T(N_q(x_0))$ is the unit tangent bundle of $N_q(x_0)$. Then $g$ is lower semi-continuous. Hence by Lemma 3 let $\epsilon$ be a positive continuous function on $\Omega_0$ such that $|df(X)-X|<1$ for $X\in T(N_{\epsilon(f)}(x_0))$ and $f\in\Omega_0$.

Let $\gamma$ be a function on $\Omega_0\times S^2$, with $\gamma(f, x)$ $C^\infty$ for each $f$ and such that $\gamma(f, x)=1$ for $x\in N_{\epsilon(f)/2}(x_0)$ and $\gamma(f, x)=0$ for $x$ in the complement of $N_{\epsilon(f)}(x_0)$.

Define $S_v$: $\Omega_0\to\Omega_0$ by

$$S_v(f)(x) = (1-v)f(x) + v[\gamma(f, x)x + (1-\gamma(f,x))f(x)].$$

Then $S_0(f)=f$ and $S_1(f)$ agrees with the identity on $N_{\epsilon(f)/2}(x_0)$.

Let $p$: $S^2-x_0\to E^2$ be stereographic projection using $x_0$ as pole with $E^2$ being the $x-y$ plane of $E^3$. Let $I_M^2$ denote the square in $E^2$ with center (0, 0) and with sides of length $M$ which are parallel to the axes. Let $D_M$: $I_M^2\to I_{\mathrm{I}}^2$ be the obvious canonical diffeomorphism onto.

Let $M$ be a positive continuous function on $\Omega_0$ such that $p^{-1}(\text{exterior } I^2_{M(f)})\subset N_{\epsilon(f)/2}(x_0)$. Let $\mathfrak{F}$ be the space of diffeomorphisms of $E^2$ which are the identity in a neighborhood of $I_{\mathrm{I}}^2$ and outside $I_{\mathrm{I}}^2$. Then let $G_v$: $\mathfrak{F}\to\mathfrak{F}$ be as in Theorem 4. Let a homotopy $F_v$: $\Omega_0\to\Omega_0$ be defined by

$$\begin{aligned} F_v(f)(x) &= b^{-1}[G_v(bfb^{-1})]b(x), \qquad b = D_M p, \qquad && x\notin N_{\epsilon(f)/2}(x_0),\\ F_v(f)(x) &= f(x) && x\in N_{\epsilon(f)/2}(x_0). \end{aligned}$$

Let $\beta(v)$ be a $C^\infty$ function which is 0 in a neighborhood of 0 and 1 in a neighborhood of 1/2. Let $\gamma(v)$ be a $C^\infty$ function which is 0 in a neighborhood of 1/2 and 1 in a neighborhood of 1. Then we define our desired contraction $T_v$: $\Omega_0\to\Omega_0$ by

$$\begin{aligned} T_v &= S_{\beta(v)} \qquad && 0\leqq v\leqq 1/2,\\ T_v &= F_{\gamma(v)} && 1/2\leqq v\leqq 1. \end{aligned}$$

The following amplifies Theorem A.

THEOREM 6. *The space $\Omega$ of all orientation preserving diffeomorphisms of $S^2$ has as a deformation retract the rotation group $SO(3)$. In fact there is a homotopy $H_v$: $\Omega\to\Omega$ such that for each $f\in\Omega$*

(a) *$H_v(f)(x)$ is $C^\infty$ in $(v, x)$,*

(b) *$H_0(f)=f$,*

(c) *$H_1(f)$ is a rotation of $S^2$, and*

(d) *if $f\in SO(3)$, $H_v(f)=f$.*

PROOF. Define $\overline{\Omega}$ as the subspace of $\Omega$ with the property that for $f\in\overline{\Omega}$, $e^1(f)=df_{x_0}(e_1)$ and $e^2(f)=df_{x_0}(e_2)$ are orthonormal. We first show that $\overline{\Omega}$ is a deformation retract of $\Omega$.

If $f\in\Omega$, let $v_0$ be the unit vector perpendicular to $e^1(f)$ in $S_{f(x_0)}$ the tangent space of $S^2$ at $f(x_0)$, and $u_0$ be $e^1(f)$ normalized. Then define

$$\overset{1}{e_t}(f) = (1 - t)\overset{1}{e}(f) + tu_0,$$
$$\overset{2}{e_t}(f) = (1 - t)\overset{2}{e}(f) + tv_0$$

Let $g_t$ be the linear transformation which sends $(e^1(f), e^2(f))$ into $(e_t^1(f), e_t^2(f))$. Let $p$: $S^2 \rightarrow S_{f(x_0)}$ be the natural projection. By Lemma 3, choose a positive continuous function $\epsilon$ on $\Omega$ with $\epsilon(f) < 1$, such that for a unit tangent vector $X$ in the tangent space of $N_{\epsilon(f)}(f(x_0))$, $d(p^{-1}g_t p)(X)$ and $X$ are independent.

Let $\gamma$ be a function on $\Omega \times S^2$ such that for each $f \in \Omega$, $\gamma(f, x)$ is $C^\infty$, is zero outside $N_{\epsilon(f)}(f(x_0))$ and is 1 on $N_{\epsilon(f)/2}(f(x_0))$. Let $p$ induce an affine structure on the hemisphere of $S^2$ with center $f(x_0)$. Define $G_v$: $\Omega \rightarrow \Omega$ by

$$G_v(f)(x) = \gamma(f, x)p^{-1}g_v p(x) + (1 - \gamma(f, x))x \cdot$$

Then $G_v$ retracts $\Omega$ onto $\Omega$.

We now define a retraction of $\bar{\Omega}$ onto $SO(3)$. For each $f \in \bar{\Omega}$ let $\alpha(f) \in SO(3)$ be the rotation sending $(f(x_0), df(e_1), df(e_2))$ into $(x_0, e_1, e_2)$. Then define $K_v$: $\bar{\Omega} \rightarrow \bar{\Omega}$ by $K_v(f) = T_v(f \circ \alpha(f))\alpha(f)^{-1}$ where $T_v$ is as in Theorem 5.

The desired homotopy $H_v$: $\Omega \rightarrow \Omega$ is obtained by composing $G_v$ and $K_v$ as in the proof of the previous theorem.

## References


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