A REMARK ON A THEOREM OF BEURLING AND HELSON

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The following result is due to Beurling and Helson [1, 2]: let μ be a finite regular Borel measure on a locally compact abelian group G, and let $\mu = \sum a_n \mu_{g_n} + \nu$ be its Lebesgue decomposition into discrete and continuous parts,[1] where μ_g is the mass 1 at g. Then if μ has unimodular Fourier-Stieltjes transform (i.e., $|\hat{\mu}| = 1$), $\sum |a_n|^2 = 1$.

The purpose of this note is to point out an extremely direct proof of the result, which also yields additional information. Let μ^* represent, as usual, the measure defined by setting $\mu^*(f) = \int f(-g)\bar{\mu}(dg)$, $f \in C_0(G)$ or, alternatively, by setting $\mu^*(E) = \bar{\mu}(E^{-1})$ for all Baire sets E. Then $\mu^* = \sum \bar{a}_n \mu_{-g_n} + \nu^*$, where ν^* is again continuous, and $(\mu^*)^\frown = \hat{\mu}^-$. Thus $|\hat{\mu}| = 1$ implies $(\mu * \mu^*)^\frown = |\hat{\mu}|^2 = 1 = \hat{\mu}_0$, and $\mu * \mu^* = \mu_0$. But

$$\mu * \mu^* = \sum_{g \in G} \left(\sum_{g_n - g_m = g} a_n \bar{a}_m \right) \mu_g + \nu'$$

where ν' is continuous since the continuous measures form an ideal in the algebra of all finite measures. Consequently

$$\mu_0 = \sum_{g \in G} \left(\sum_{g_n - g_m = g} a_n \bar{a}_m \right) \mu_g \qquad \text{(and } \nu' = 0\text{)},$$

and we obtain

$$\sum_{g_n - g_m = g} a_n \bar{a}_m = \delta_{0,g},$$

in the usual notation, which contains the Beurling-Helson result as the special case $g = 0$. Furthermore, setting $\mu^d = \sum a_n \mu_{g_n}$, the fact that $\mu_0 = \mu^d * \mu^{d*}$ shows that the discrete part of μ also has a unimodular transform: for $|\hat{\mu}^d|^2 = \hat{\mu}_0 = 1$ and thus $|\hat{\mu}^d| = 1$.

References

1. H. Helson, *Isomorphisms of abelian group algebras*, Ark. Mat. vol. 2 (1953) pp. 475–487.

2. W. F. Eberlein, *A note on Fourier-Stieltjes transforms*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 310–312.

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[1] That is, ν vanishes on all one point sets, while $\sum a_n \mu_{g_n}$ is a (countable) sum converging in the usual norm.