A NOTE ON PIERCING A DISK

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In this note we give a sufficient condition that a 2-cell or disk in E^3 be pierced by a tame arc at each point of its interior. This condition is simply that each arc in the disk be tame. The method of proof leans heavily upon some results of Bing and Moise as well as upon a theorem on the union of tame disks by one of the present authors.

REMARK. Lemma 7 of [2] asserts that if two tame disks in E^3 intersect in a simple arc on the boundary of each, then the union of the two disks is a tame disk. The proof of this result is an application of Theorem 5.1 of [4]. It is then a simple matter to see that if D is a disk in E^3 having an interior which is triangulated in a locally finite manner into tame 2-cells, then D is locally tame at each of its interior points. If in addition D has a tame boundary, then D is tame by either Theorem 5.13 of [5] or Theorem 5.1 of [4]. These observations will be used in our proof.

THEOREM. *Let D be a disk in E^3 such that each arc in D is tame. Then D is pierced by a tame arc at each point of D^0 (interior of D).*

PROOF. Let q be any point in D^0. Let R be the plane rectangle $\{(x, y) \mid -1 \leqq x \leqq 1,\ -1 \leqq y \leqq 1\}$ and let h be a homeomorphism of R onto D. There is no loss of generality in assuming that $h^{-1}(q) = (0, 0)$. We construct an infinite cell decomposition of R by means of the line segments

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\begin{aligned}
J_0 &= \{(x, y) \mid x = 0, -1 \leqq y \leqq -1\},\\
K_0 &= \{(x, y) \mid 0 \leqq x \leqq 1, y = 0\},\\
J_n^- &= \{(x, y) \mid x = 1/n, -1 \leqq y \leqq 1; n > 1\},\\
J_n^+ &= \{(x, y) \mid x = (n-1)/n, -1 \leqq y \leqq 1; n > 2\},\\
K_n^- &= \{(x, y) \mid 0 \leqq x \leqq 1, y = -1/2 - (-1)^n(n-1)/2n; n > 1\},\\
K_n^+ &= \{(x, y) \mid 0 \leqq x \leqq 1, y = 1/2 - (-1)^n(n-1)/2n; n > 1\}.
\end{aligned}
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Denote the union of all of these segments together with the closure of this union by G'. The grid G' decomposes R into countably many closed disks and an isomorphic decomposition of D is induced through the homeomorphism h.

Let $R_1 = \{(x, y) \mid 0 \leqq x \leqq 1,\ -1 \leqq y \leqq 1\}$ and $R_2 = \overline{R} - (R_1 - J_0)$. Denote $h(R_i)$ by D_i, $i = 1, 2$, and $h(G')$ by G. (Clearly, each arc in G is

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tame by our assumption on D.) Define the real-valued function $f(x)$ on D_1 by setting $f(x)=2^{-1}\rho_1(x)$ where $\rho_1(x)$ is the distance from the point x to the compact set $G\cup D_2$. Using the Bing Approximation Theorem [1], we choose a homeomorphism g of D_1 into E^3 such that (i) $g(D_1)$ is locally polyhedral except on G and (ii) $\rho[x, g(x)]\leqq f(x)$. Letting $g(D_1)=D_1'$, we observe that repeated application of the Remark above together with the construction of G implies that D_1' has a tame boundary and is locally tame in its interior. Hence in view of the Remark, D_1' is tame.

We next "swell" D_1' into a tame 3-cell e^3 which is locally polyhedral except on the boundary of D_1 and which is such that $(D_1')^0$ lies in $(e^3)^0$ while $e^3\cap D_2$ is precisely the tame arc $h(J_0)$. This is done by making use of Lemma 5.1 of [3] and Theorem 9.3 of [4]. We denote the boundary of e^3 by S and observe that the boundary of D_1 separates S into two hemispheres, say S_1 and S_2.

Let $\{C_i\}$ be a sequence of all disks in the decomposition of D_1 by G for which $S_1\cap C_i$ is not empty. Each intersection $S_1\cap C_i$ is a compact set in the interior C_i^0 of C_i because the hemisphere S_1 is disjoint from the grid G except along the boundary of D_1. Thus, for each i, the distance $\rho(S_1\cap C_i,\ S_1\cap D_1-C_i)$ is positive. Appealing to the Zoretti Theorem [6] we note that each component $K_{i\alpha}$ of $S_1\cap C_i$ may be enclosed in a simple closed curve $A_{i\alpha}$ in S_1^0 with the property that $A_{i\alpha}\cap(S_1\cap C_i)$ is empty and $A_{i\alpha}$ is as close to $K_{i\alpha}$ as we may wish. There is no loss of generality in assuming $A_{i\alpha}$ to be polygonal. Hence for each i we may choose a finite number of the curves $A_{i\alpha}$ and so obtain a finite number of polyhedral disks C_{ij}' in S_1^0 whose interiors cover $S_1\cap C_i$.

To give the desired construction, we choose the finite number of simple closed curves A_{1j} in such a way that $\rho(A_{1j},\ S_1\cap C_1) <2^{-1}\rho(S_1\cap C_1,\ S_1\cap D_1-C_1)$. Let the union of the corresponding disks C_{1j}' be the polyhedral set C_1'. Then we choose the curves A_{2j} in such a way that $\rho(A_{2j},\ S_1\cap C_2)$ is less than one half of the minimum of $\rho(S_1\cap C_2,\ S_1\cap D_1-C_2)$ and $\rho(C_1',\ S_1\cap D_1-C_1)$. In this way we obtain a polyhedral set C_2' in S_1^0 such that $C_1'\cap C_2'$ and $C_2'\cap\overline{D}_1-(C_2-J_0)$ are both empty. Continuing this procedure in the obvious way, we obtain a sequence $\{C_i'\}$ of disjoint closed polyhedral sets in S_1^0 such that the union $\bigcup(C_i')^0$ contains $S_1^0\cap D_1$. By simply ordering the components of each set C_i' we obtain a sequence $\{C_i''\}$ of connected closed disjoint polyhedral sets in S_1^0 covering $S_1^0\cap D_1$. We note that the method of construction implies that the diameters of the sets C_i'' approach zero as i increases indefinitely.

Once again the Zoretti Theorem may be applied to enclose each set C_i'' in a simple closed curve A_i in S_1^0, each point of A_i being as close to C_i'' as desired. Again we assume each A_i to be polygonal. A construction almost identical to that above may be described inductively so as to yield a sequence of polygonal simple closed curves $\{A_i\}$ such that $A_i \cap A_j$ is empty if $i \neq j$, each A_i bounds a polyhedral disk P_i in S_1^0 with C_i'' lying in P_i^0 and as i increases without bound, the diameter of P_i approaches zero. However it may happen in this construction that for $i \neq j$ we have P_i contained in P_j^0. So we let $\{P_j'\}$ be a subsequence selected from $\{P_i\}$ such that $P_j' \cap P_k'$ is empty if $j \neq k$ and such that for each i there is a j such that P_i lies in P_j'. Clearly we still have the fact that $\bigcup(S_1^0 \cap C_i)$ is contained in $\bigcup P_j'$.

Let x be any point in $S_1^0 - \bigcup P_j'$. Since S_1 is polyhedral except on its boundary, there is a tame arc T in S_1 with endpoints x and q and such that $T-q$ lies in S_1^0. We may choose T to be polygonal except at q. In general T will intersect some of the disks in the sequence $\{P_j'\}$. We construct an arc T_∞ having the same properties as T but which does not meet the interior of any disk P_j'. To do this, we describe a sequence of arcs T_i as follows: If T fails to meet the interior of P_1', we let $T_1 = T$. Otherwise, let T be parametrized by way of a homeomorphism p of the unit interval where $p(0) = x$ and $p(1) = q$. There will be a first and a last point in which T intersects P_1'. We replace the corresponding subarc on T by the shorter of the two arcs on the boundary A_1' of P_1' having the same endpoints and call the resulting arc T_1. Suppose that T_j has been defined so that it is an arc from x to q, that $T_j - q$ lies in S_1^0 and $T_j \cap \bigcup_{n=1}^{j} (P_n')^0$ is empty. The arc T_{j+1} is T_j if $T_j \cap (P_{n+1}')^0$ is empty; otherwise we proceed as was done in obtaining T_1 from T. That is, T_j is parametrized and a subarc of T_j is replaced by an arc in A_{j+1}', the boundary of P_{j+1}'. It is easily shown that the sequence of arcs $\{T_j\}$ converges to an arc T_∞ since the diameters of the disks P_j' approach zero with increasing j. By construction T_∞ is disjoint from the sets C_i'' and lies in S_1^0 except at q. It follows that T_∞ meets the disk D_1 only at q. Also since S_1 is disjoint from D_2 except along the arc $h(J_0)$ in the boundary of S_1, T_∞ is disjoint from the disk D_2 except at the point q.

By an identical construction an arc T_∞' can be obtained on the other hemisphere S_2 such that $T_\infty' \cap D = q$. The union $A = T_\infty \cup T_\infty'$ is a tame arc since it lies on a tame 2-sphere S and A meets D only at q. Thus A pierces D at q and the proof is complete.

The following results are now obvious.

COROLLARY 1. *If a disk D in E^3 has the property that each arc in D^0 is tame, then D may be pierced by a tame arc at each point of D^0.*

COROLLARY 2. *A disk in E^3 which is pierced by no tame arc has a wild arc in each of its open sets.*

We believe the converse of Corollary 1 to be true also but have not been able to prove it. In this connection we may call attention to a conjecture of O. G. Harrold which asserts that if D is a disk in E^3 such that all arcs in D are tame, then D itself is tame. Our result lends some support to this conjecture.

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