A NOTE ON THE ACTION OF $SO(3)$

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1. Introduction. The main purpose of this note is a proof of the following theorem about the action of the 3-dimensional rotation group, $SO(3)$, as a topological transformation group.

THEOREM. *If $(SO(3), C^n)$ denotes $SO(3)$ operating on a closed n-cell, then the orbit space $C^n/SO(3)$ is acyclic over the integers; that is, $H^i(C^n/SO(3); Z)=0$ for $i>0$.*

This theorem has been proved for finite groups [6] and for toral groups [5]. It might be conjectured for all compact Lie groups.

We shall denote a transformation group by (G, X) and the natural map of X onto the orbit space X/G by

$$\pi: X \to X/G.$$

For any point $x\in X$, we denote by $H_x\subset G$ the isotropy subgroup of G at $x\in X$. The group G will be a compact Lie group. The space X is assumed to be locally compact, locally connected, finite dimensional and separable metric. We shall use the Alexander-Wallace-Spanier cohomology groups. A subscript c will denote cohomology with compact supports. The coefficient ring J will denote either the integers Z, or the integers mod 2, Z_2.

2. **The double coset spaces of $SO(3)$.** We shall need certain basic facts about the double coset spaces of the 3-dimensional rotation group.

LEMMA 1. *Let $N\subset SO(3)$ be the normalizer of a maximal torus S^1. If $H\subset SO(3)$ is a closed subgroup, then the double coset space of $SO(3)$ with respect to N and H, $(SO(3)/N)/H$, is described as follows:*

(i) dim $H=3$ *and $(SO(3)/N)/H$ is a point,*

(ii) dim $H=1$ *and $(SO(3)/N)/H$ is an arc,*

(iii) *H is finite of odd order, and $(SO(3)/N)/H$ is a real projective plane,*

(iv) *H is finite of even order, and $(SO(3)/N)/H$ is a closed 2-cell.*

We shall only consider the last two cases. The space $SO(3)/N$ is a real projective plane. If we consider the finite transformation group $(H, SO(3)/N)$, then $(SO(3)/N)/H$ is an nonorientable closed 2-manifold, or it is a closed 2-cell. We shall use this as a known fact [1].

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Now $(SO(3)/N)$ is acyclic for any coefficient field save the integers mod 2. If H is a subgroup of odd order, H is cyclic, thus H has a fixed point, but from Smith's result the fixed point set of any p-subgroup of H consists of one point [11], thus H has only one fixed point and $(SO(3)/N)/H$ is a projective plane.

If H contains an element of order 2, then the fixed point set of this transformation of order 2, $(Z_2, SO(3)/N)$ cannot consist of one point [4]. Since the fixed point set must have Euler characteristic 1, it follows that it consists of a point and a collection of disjoint simple closed curves. Since

$$\pi_*: \pi_1(SO(3)/N) \to \pi_1((SO(3)/N)/Z_2)$$

is onto [7], the orbit space $(SO(3)/N)/Z_2$ is a closed 2-cell. We may factor π into

$$SO(3)/N \xrightarrow{\alpha} (SO(3)/N)/Z_2 \xrightarrow{\beta} (SO(3)/N)/H$$

where β is a ramified cover, so $(SO(3)/N)/H$ must also be a closed 2-cell.

We have, then, briefly characterized some double coset spaces of $SO(3)$.

3. **A factorization of the map π.** If $(SO(3), X)$ is an operation of $SO(3)$, then let $N \subset SO(3)$ be the normalizer of a maximal torus S^1 and we may factor

$$\pi: X \to X/SO(3)$$

into maps

$$\alpha: X \to X/N,$$
$$\beta: X/N \to X/SO(3)$$

where $\beta\alpha = \pi$ and X/N is the orbit space of the transformation group (N, X). If $y \in X/SO(3)$, then $\beta^{-1}(y)$ is topologically the double coset space $(G/H_x)/N$, where $\pi(x) = y$.

A slice [9] $K_x \subset X$ is a closed connected subset of X satisfying

1. $x \in K_x$,
2. $gK_x \cap K_x \neq \emptyset \Leftrightarrow g \in H_x$,
3. $H_y \subset H_x$ for $y \in K_x$,
4. $SO(3)(K_x)$ is a closed neighborhood of the orbit $SO(3)(x)$.

Let $B(x) = \alpha(G(x))$, and let $S_x = \alpha(SO(3)(K_x))$, then S_x is a closed neighborhood of $B(x)$. We may define a retraction $r: S_x \to B(x)$ by $r\alpha(gy) = \alpha(gx)$ for $y \in K_x$. We also define for $y \in K_x$ a map $k_y: SO(3)/H_y \to S_x$ by $k_y(gH_y) = \alpha(gy)$. Since $k_y(gH_y) = k_y(g_1H_y)$ if and only if there is a $t\in N$ such that $g_1^{-1}tg\in H_y$, the natural map $\pi_y\colon SO(3)/H_y \to (SO(3)/H_y)/N$ induces a homeomorphism $\bar{k}_y\colon (G/H_y)/N\to B(y)$ such that the diagram

$$\begin{array}{cc} (1) & \begin{array}{ccccc} SO(3)/H_y & \xrightarrow{k_y} & S_x & \xrightarrow{r} & B(x) \\ \downarrow \pi_y & & \uparrow i_y & & \\ (SO(3)/H_y)/N & \xrightarrow{\bar{k}_y} & B(y) & & \end{array} \end{array}$$

is commutative. Since $H_y\subset H_x$ there is also induced a commutative diagram

$$\begin{array}{ccc} SO(3)/H_y & \xrightarrow{M_{x,y}} & SO(3)/H_x \\ \downarrow \pi_y & & \downarrow \pi_x \\ (SO(3)/H_y)/N & \xrightarrow{\overline{M}_{x,y}} & (SO(3)/H_x)/N. \end{array}$$

Note in particular, $\bar{k}_x\overline{M}_{x,y}=ri_y\bar{k}_y$.

Define

$$\sigma_y\colon SO(3)/N\to (SO(3)/H_y)/N$$

by $\sigma_y(gN)=\pi_y(N_g^{-1}H_y)$, then the maps $i_y\bar{k}_y\sigma_y\colon SO(3)/N\to S_x$ are all homotopic.

A transformation group $(SO(3),\ X)$ is said to be almost free if H_x has no element of order 2 for any $x\in X$. It is quite easy to check in this case that

$$\overline{M}_{x,y}^*\colon H^*((SO(3)/H_x)/N;J)\cong H^*((SO(3)/H_y)/N;J),$$
$$\sigma_y^*\colon H^*((SO(3)/H_y)/N;J)\cong H^*(SO(3)/N;J).$$

Since $\bar{k}_y^*i_y^*r^*=\overline{M}_{x,y}^*\bar{k}_x^*$ is an isomorphism, it follows that $i_y^*\colon H^*(S_x;J)\to H^*(B(y);J)$ is onto for $y\in K_x$. Since $\sigma_y^*\bar{k}_y^*i_y^*$ is independent of y, the kernel of

$$i_y^*\colon H^*(S_x;J)\to H^*(B(y);J)$$

is independent of $y\in K_x$. Thus, for an almost free transformation group $(SO(3),X)$, every value of the map $\beta\colon X/N\to X/SO(3)$ is regular in the sense of Fary [8]. The map, $\beta\colon X/N\to X/SO(3)$, induces on $X/SO(3)$ a coefficient sheaf for each $i\geqq 0$, where the stalk is $H^i(\beta^{-1}(y);J)$. Since β is a regular map, the coefficient sheaf is locally constant. Furthermore, since $H^i(\beta^{-1}(y);\ J)\cong Z_2$ or 0 for $i>0$, the induced coefficient sheaf is constant.

THEOREM 1. *If $(SO(3),\ X)$ is an almost free transformation group, there is a spectral sequence $\{E_r^{s,t}\}$ such that*

$$E_2^{s,t} \cong H_c^s(X/SO(3); H^t(SO(3)/N; J))$$

and whose E_∞-term is associated with $H_c^*(X/N; J)$.

This is, of course, the spectral sequence of the mapping [**2**, p. 21–01]

$$\beta: X/N \rightarrow X/SO(3).$$

4. **Application to actions of $SO(3)$ on a compact space.** If $(SO(3), X)$ is an action on a compact space, let

$$U = \{x/x \in X, H_x \text{ is a finite group of odd order}\}.$$

The set U is open, since for any point $x \in X$, there is an open set V_x such that H_y is conjugate to a subgroup of H_x for $y \in V_x$ [**10**, p. 241]. Let

$$S = \alpha(X - U),$$
$$T = \pi(X - U).$$

We may identify the compact cohomology of U/N and $U/SO(3)$ respectively with the relative cohomology of the pairs $(X/N, S)$ and $(X/SO(3), T)$. Since $(SO(3), U)$ is almost free, we have

THEOREM 2. *If $(SO(3), X)$ denotes an operation of $SO(3)$ on a compact space X, there is an exact sequence*

$$\rightarrow H^i(X/SO(3), T; Z_2) \xrightarrow{d} H^{i+3}(X/SO(3), T; Z) \xrightarrow{\beta^*} H^{i+3}(X/N, S; Z)$$
$$\rightarrow H^{i+1}(X/SO(3), T; Z_2) \rightarrow.$$

The spectral sequence in Theorem 1 has, for $J = Z$, just two fibre degrees 0 and 2, since $H^0(G/N; Z) = Z$, $H^2(G/N; Z) \cong Z_2$ and $H^i(G/N; Z) = 0$ otherwise. As shown in [**3**, p. 328] we may associate with every spectral sequence having two fibre degrees a Gysin sequence as we have done in Theorem 2. Now we can prove the theorem announced in the beginning of this note.

If $(SO(3), C^n)$ is an operation of $SO(3)$ on a closed n-cell, then $H^i(C^n/N; J) = 0$, $i > 0$ as shown in [**5**]. By the Vietoris mapping theorem, $\beta^*: H^i(T; J) \cong H^i(S; J)$ for all $i > 0$. If we consider the diagram

$$\text{(2)} \quad \begin{array}{ccccc} \rightarrow H^{i-1}(S; J) & \xrightarrow{\delta^*} & H^i(C^n/N, S; J) & \longrightarrow & H^i(C^n/N; J) \rightarrow \\ \uparrow \beta^* & & \uparrow \beta^* & & \uparrow \beta^* \\ H^{i-1}(T; J) & \xrightarrow{\delta^*} & H^i(C^n/SO(3), T; J) & \rightarrow & H^i(C^n/SO(3); J) \rightarrow \end{array}$$

we see immediately that

$$\beta^*: H^i(C^n/SO(3), T; J) \to H^i(C^n/N, S; J)$$

is onto for $i \geqq 0$. Since $C^n/SO(3)$ is finite dimensional, let m be the largest integer such that $H^m(C^n/SO(3), T; Z_2) \neq 0$, then from the spectral sequence in Theorem 1 it follows by the maximal cocycle argument that $H^{m+2}(X/N, S; Z_2) \neq 0$, which implies

$$H^{m+2}(X/SO(3), T; Z_2) \neq 0,$$

a contradiction. Therefore

$$H^i(X/SO(3), T; Z_2) = 0 \qquad \text{for } i \geqq 0.$$

By the exact sequence of Theorem 2, it follows that

$$\beta^*: H^i(X/SO(3), T; Z) \cong H^i(X/N, S; Z).$$

If we apply the "five-lemma" to the commutative diagram (2), we conclude that

$$\beta^*: H^i(X/SO(3); Z) \cong H^i(X/N; Z)$$

and X/N is acyclic.

This concludes the proof of the main result. In another note we shall show that $C^n/SO(3)$ is an absolute retract (AR), but this requires a more extensive investigation of local properties of $C^n/SO(3)$.

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