INEQUALITIES FOR POLYNOMIALS[1]

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A.1. **Introduction.** Let $p(z)$ be a polynomial of degree n such that $|p(z)| \leqq 1$ in the unit disc $|z| \leqq 1$. The following results are known.

THEOREM A. *For* $|z| = 1$, $|p'(z)| \leqq n$.

THEOREM B. *For* $|z| = R > 1$, $|p(z)| \leqq R^n$.

THEOREM C. *If*

$$[\max |p(z)|, |z| = 1] = 1,$$

then

$$[\max |p(z)|, |z| = \rho < 1] \geqq \rho^n.$$

Equality in Theorems A, B and C occurs only for $p(z) = \alpha z^n$, where $|\alpha| = 1$. Theorem A is a consequence of *S*. Bernstein's theorem on the derivative of a trigonometric polynomial (for references see [**7**] or [**2**, pp. 206, 213]). Theorem B is a deduction from the maximum principle (see [**6**, vol. 1, p. 137, Problem III, p. 269]). Theorem C has been proved by E. H. Zarantonello (see [**8**, pp. 44–45]). It can be proved more briefly by applying the maximum modulus principle to $z^n p(z^{-1})$ in the region $|z| > 1$.

If $p(z)$ has all its zeros in $|z| \geqq 1$, Theorems A and B can be sharpened.

THEOREM D. *For* $|z| = 1$, $|p'(z)| \leqq n/2$.

THEOREM E. *For* $|z| = R > 1$, $|p(z)| \leqq (1 + R^n)/2$.

Theorem D was conjectured by Erdös and proved by Lax [**5**]; for other proofs see [**4**] and [**3**]. Theorem E was deduced from Theorem D by Ankeny and Rivlin [**1**] and later generalized by Boas (see [**3**, Theorem 1]).

In analogy with Theorem C we prove

THEOREM 1. *Let $p(z)$ be a polynomial of degree n such that*

$$[\max |p(z)|, |z| = 1] = 1.$$

If $p(z)$ has all its zeros in $|z| \geqq 1$ and at least one in $|z| > 1$, then there exists a positive number δ such that

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$$[\max |p(z)|, |z| = \rho] > (1 + \rho^n)/2,$$

for $(1-\delta) < \rho < 1$.

By applying Theorem E to $z^n p(z^{-1})$ we can deduce that if $p(z)$ has all its zeros on $|z| = 1$ and $|p(z)| \leqq 1$ for $|z| \leqq 1$ then

$$|p(z)| \leqq (1 + \rho^n)/2$$

for $|z| = \rho < 1$ and the conclusion of Theorem 1 cannot hold.

As pointed out by Ankeny and Rivlin [1] the example $6^{-1}(z+1/2) \cdot (z+3)$ shows that the converse of Theorem E is not true. They however proved a result equivalent to the following [1, Theorem 3].

THEOREM F. *If $p(z)$ is a polynomial of degree n such that*

$$[\max |p(z)|, |z| = 1] = 1,$$

and

$$[\max |p(z)|, |z| = R] \leqq (1 + R^n)/2,$$

for $1 < R < 1+\delta$, where δ is any positive number, then $p(z)$ does not have all its zeros within the unit circle.

We observe that the hypothesis in Theorem F can be weakened without affecting the conclusion. In fact, we have

THEOREM 2. *If $p(z)$ is a polynomial of degree n such that*

$$[\max |p(z)|, |z| = 1] = 1,$$

and

$$[\max |p(z)|, |z| = R_m > 1] \leqq (1 + R_m^n)/2,$$

for a sequence $R_m(\downarrow 1)$, then $p(z)$ does not have all its zeros within the unit circle.

In §4 we show that Theorems A and B can be deduced from Theorems D and E respectively, in a very simple way. Although the proof of Theorem E as given by Ankeny and Rivlin makes use of Theorem B the one given by Boas [3] is independent of both Theorems A and B.

2. LEMMA. *Let $p(z)$ be a polynomial of degree n such that*

$$[\max |p(z)|, |z| = 1] = 1.$$

If $p(z)$ has all its zeros in $|z| \leqq 1$ and at least one in $|z| < 1$, then there exists a positive number δ such that

$$[\max |p(z)|, |z| = R] > (1 + R^n)/2$$

for $1 < R < 1 + \delta$.

PROOF OF THE LEMMA. If the result is false then there exists a sequence $R_1, R_2, \cdots, R_m, \cdots$ ($R_m \downarrow 1$) such that

$$[\max |p(z)|, |z| = R_m > 1] \leqq (1 + R_m^n)/2.$$

Let $p(z)$ attain its maximum on the unit circle at $z = e^{i\alpha}$ and suppose that $p(e^{i\alpha}) = e^{i\beta}$. Let $z_1, z_2, \cdots, z_m, \cdots$ ($z_m \to e^{i\alpha}$) be the intersection[2] $\{p^{-1}(Re^{i\beta}), R > 1\} \cap \{|z| = R_m, m = 1, 2, \cdots\}$. Then

$$\begin{aligned} |p'(e^{i\alpha})| &= \left| \lim_{m\to\infty} \frac{p(z_m) - p(e^{i\alpha})}{z_m - e^{i\alpha}} \right| \\ &\leqq \lim_{m\to\infty} \frac{2^{-1}(1 + R_m^n) - 1}{R_m - 1} \qquad (1) \\ &= n/2. \end{aligned}$$

But according to Walsh's generalization of Laguerre's theorem [9, Lemma 1, p. 13]

$$\frac{p'(e^{i\theta})}{p(e^{i\theta})} = \frac{n}{e^{i\theta} - w},$$

for points $e^{i\theta}$ other than zeros of $p(z)$ and $|w| < 1$. Hence $|e^{i\theta} - w| < 2$ and

$$\left| \frac{p'(e^{i\theta})}{p(e^{i\theta})} \right| > n/2.$$

We conclude that (1) cannot hold. This proves the lemma.

3. PROOF OF THEOREM 1. Consider $P(z) = z^n p(z^{-1})$ which is a polynomial of degree n for which

$$[\max |P(z)|, |z| = 1] = 1.$$

Further, all the zeros of $P(z)$ lie in $|z| \leqq 1$ with at least one in $|z| < 1$. By the lemma it follows that there exists a positive number δ' such that

$$[\max |P(z)|, |z| = R] > (1 + R^n)/2$$

for $1 < R < 1 + \delta'$. For $1 < R < 1 + \delta'$ we have therefore

[2] In case $\{p^{-1}(Re^{i\beta}), R > 1\}$ consists of a number of different arcs we may select any one of these arbitrarily.

$$[\max |z^n p(z^{-1})|, |z| = R] = [\max R^n |p(z^{-1})|, |z| = R] > (1 + R^n)/2$$

or

$$[\max |p(z^{-1})|, |z| = R] > \frac{1}{2}\left\{\left(\frac{1}{R}\right)^n + 1\right\}$$

or

$$\left[\max |p(z)|, |z| = \frac{1}{R}\right] > \frac{1}{2}\left\{\left(\frac{1}{R}\right)^n + 1\right\}.$$

Replacing $1/R$ by ρ in above we conclude that there exists a positive number δ such that

$$[\max |p(z)|, |z| = \rho] > (1 + \rho^n)/2$$

for $(1-\delta) < \rho < 1$.

Proof of Theorem 2. The assumption that $p(z)$ has all its zeros in $|z| < 1$ contradicts the lemma.

4. **Deduction of Theorems A and B from Theorems D and E respectively.** Let $p(z)$ be a polynomial of degree n such that $|p(z)| \leqq 1$ in the unit disc $|z| \leqq 1$. For a fixed $R > 1$ let

$$[\max |p(z)|, |z| = R > 1] = M_p(R) = |p(Re^{i\alpha})|,$$

where

$$p(Re^{i\alpha}) = M_p(R)e^{i\beta}.$$

Consider the function $F(z) = \{e^{i\beta} + p(z)\}/2$ which is a polynomial of degree n having no zeros in $|z| < 1$. Since $|F(z)| \leqq 1$ in the unit disc $|z| \leqq 1$, it satisfies the conditions of Theorems D and E. Consequently

(i) $$[\max |F'(z)|, |z| = 1] = \left[\max \left|\frac{p'(z)}{2}\right|, |z| = 1\right] \leqq n/2,$$

or

$$[\max |p'(z)|, |z| = 1] \leqq n;$$

$$[\max |F(z)|, |z| = R] = \left[\max \left|\frac{e^{i\beta} + p(z)}{2}\right|, |z| = R\right]$$

(ii) $$= \frac{1 + M_p(R)}{2} \leqq (1 + R^n)/2,$$

or

$$[\max |p(z)|, |z| = R > 1] = M_p(R) \leqq R^n.$$

B. In this section we give analogous results for the mean value of $|p(z)|$. If $p(z)$ is a polynomial of degree n then [2, pp. 211, 98]

$$\int_0^{2\pi} |p'(e^{i\theta})|\,d\theta \leqq n\int_0^{2\pi} |p(e^{i\theta})|\,d\theta, \tag{2}$$

$$\int_0^{2\pi} |p(Re^{i\theta})|\,d\theta \leqq R^n\int_0^{2\pi} |p(e^{i\theta})|\,d\theta, \qquad R > 1. \tag{3}$$

Clearly

$$\int_{|z|=R>1} |z^n p(z^{-1})|\,dz \geqq \int_{|z|=1} |z^n p(z^{-1})|\,dz = \int_0^{2\pi} |p(e^{i\theta})|\,d\theta,$$

or

$$\int_{|z|=R>1} |p(z^{-1})|\,dz \geqq R^{-n}\int_0^{2\pi} |p(e^{i\theta})|\,d\theta,$$

or

$$\int_0^{2\pi} |p(\rho e^{i\theta})|\,d\theta \geqq \rho^n\int_0^{2\pi} |p(e^{i\theta})|\,d\theta, \qquad \rho < 1. \tag{4}$$

Equality in (2), (3) and (4) occurs only for $p(z) = \lambda z^n$, where λ is a constant.

If $p(z)$ has no zeros in $|z| < 1$, (2) and (3) can be sharpened.

$$\int_0^{2\pi} |p'(e^{i\theta})|\,d\theta \leqq (n\pi/4)\int_0^{2\pi} |p(e^{i\theta})|\,d\theta, \tag{5}$$

$$\int_0^{2\pi} |p(Re^{i\theta})|\,d\theta < \left\{\frac{\pi}{4}(R^n - 1) + 1\right\}\int_0^{2\pi} |p(e^{i\theta})|\,d\theta, \tag{6}$$

with equality in (5) only for $p(z) = \lambda + \mu z^n$ where $|\lambda| = |\mu|$. (5) was proved by N. G. de Bruijn [4, pp. 597–598]. (6) can be deduced from (5) as follows.

Let us assume that $p(z)$ does not have the form $\lambda + \mu z^n$, $|\lambda| = |\mu|$. For each ϕ, $0 \leqq \phi < 2\pi$, we have

$$|p(Re^{i\phi})| \leqq \int_1^R |p'(re^{i\phi})|\,dr + |p(e^{i\phi})|,$$

$$\int_0^{2\pi} |p(Re^{i\phi})|\,d\phi \leqq \int_1^R dr\int_0^{2\pi} |p'(re^{i\phi})|\,d\phi + \int_0^{2\pi} |p(e^{i\phi})|\,d\phi.$$

$p'(z)$ is a polynomial of degree $n-1$ and therefore from (3) we have

$$\int_0^{2\pi} |p(Re^{i\phi})|\,d\phi \leqq \int_1^R r^{n-1}dr \int_0^{2\pi} |p'(e^{i\phi})|\,d\phi + \int_0^{2\pi} |p(e^{i\phi})|\,d\phi$$

$$< \frac{n\pi}{4}\int_0^{2\pi} |p(e^{i\phi})|\,d\phi \int_1^R r^{n-1}dr + \int_0^{2\pi} |p(e^{i\phi})|\,d\phi,$$

from (5) since $p(z)$ has no zeros in $|z| < 1$. Consequently

$$\int_0^{2\pi} |p(Re^{i\phi})|\,d\phi < \left\{\frac{\pi}{4}(R^n - 1) + 1\right\} \int_0^{2\pi} |p(e^{i\phi})|\,d\phi.$$

If $p(z) = \lambda + \mu z^n$, where $|\lambda| = |\mu|$, then

$$\int_0^{2\pi} |p(Re^{i\phi})|\,d\phi < \left\{\frac{\pi}{4}(R^n - 1) + 1\right\} \int_0^{2\pi} |p(e^{i\phi})|\,d\phi.$$

We also prove the following theorem.

If $p(z)$ has no zeros in $|z| < 1$ then there exists a positive number δ such that

$$\int_0^{2\pi} |p(\rho e^{i\phi})|\,d\phi > \left\{1 - \frac{\pi}{4}(1 - \rho^n)\right\} \int_0^{2\pi} |p(e^{i\phi})|\,d\phi$$

for $(1-\delta) < \rho < 1$.

Let us assume that $p(z) \neq \lambda + \mu z^n$ where $|\lambda| = |\mu|$. If now the theorem is false then there exists a sequence $\rho_1, \rho_2, \cdots, \rho_m, \cdots$ $(\rho_m \uparrow 1)$ such that

$$\int_0^{2\pi} |p(\rho_m e^{i\phi})|\,d\phi \leqq \left\{1 - \frac{\pi}{4}(1 - \rho_m^n)\right\} \int_0^{2\pi} |p(e^{i\phi})|\,d\phi, \qquad (m = 1, 2, \cdots).$$

Then

$$\int_0^{2\pi} |p'(e^{i\phi})|\,d\phi = \int_0^{2\pi} \left|\lim_{m\to\infty} \frac{p(e^{i\phi}) - p(\rho_m e^{i\phi})}{e^{i\phi} - \rho_m e^{i\phi}}\right| d\phi$$

$$\geqq \lim_{m\to\infty} \frac{1}{1-\rho_m}\left[\int_0^{2\pi} |p(e^{i\phi})|\,d\phi - \left\{1 - \frac{\pi}{4}(1 - \rho_m^n)\right\} \int_0^{2\pi} |p(e^{i\phi})|\,d\phi\right]$$

$$= \frac{n\pi}{4}\int_0^{2\pi} |p(e^{i\phi})|\,d\phi,$$

which contradicts (6). Finally, if $p(z)=\lambda+\mu z^n$, $|\lambda|=|\mu|$, then

$$\int_0^{2\pi} | p(\rho e^{i\phi}) | d\phi > \left\{1 - \frac{\pi}{4}(1-\rho^n)\right\} \int_0^{2\pi} | p(e^{i\phi}) | d\phi.$$

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