# HYPERSPACES OF THE INVERSE LIMIT SPACE

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**Introduction.** Throughout the following $X$ will denote a metric continuum, $2^X$ the set of all nonempty closed subsets of $X$ and $C(X)$ the set of all nonempty subcontinua of $X$. It is the purpose of this paper to answer questions raised in [4] about the dimension and homological properties of $C(X)$ when $X$ is non-Peanian. In §1 $C(X)$ is shown to be acyclic in all dimensions and in §2 sufficient conditions for the finite dimensionality of $C(X)$ are obtained.

**Notation.** If $(U_1, \cdots, U_n)$ is a collection of subsets of a topological space $X$, then $\langle U_1, \cdots, U_n\rangle$ denotes $\{E \in 2^X \mid E \subset \bigcup_{i=1}^n U_i$ and $E \cap U_i \neq \emptyset$ for each $i\}$. If $X$ is a topological space, then the finite topology on $2^X$ is the one generated by collections of the form $\langle U_1, \cdots, U_n\rangle$ with $U_1, \cdots, U_n$ open subsets of $X$.

$C(X)$ denotes the space of all nonempty subcontinua of $X$ with the topology inherited from $2^X$ with the finite topology. If $X = \lim(X_i, f_i, I)$ where $X_i$ is a metric continuum, $f_i$ is continuous and $I$ is the set of natural numbers, then $X$ is a metric continuum. [See [1] for an explanation of this notation used in the description of the inverse limit space.] Now $C(X_i)$ is defined and we define $f_i' : C(X_{i+1}) \to C(X_i)$ by $f_i'(E) = (f_i^* \mid C(X_{i+1}))(E) = f_i(E)$, where $f_i^*: 2^{X_{i+1}} \to 2^{X_i}$ is continuous by [6, Theorem 5.10], so that $f_i'$ is continuous. Let $C_\infty(X) = \lim(C(X_i), f_i', I)$, where $C_\infty(X)$ is given the relative topology inherited from the product of the $C(X_i)$'s with the product topology. Let $\pi_n: X \to X_n$ be the projection map on $X$ and $\pi_n': C_\infty(X) \to C(X_n)$ be the projection map on $C_\infty(X)$.

## 1. Homology of $C(X)$.

First we show that $C(X)$ and $C_\infty(X)$ are homeomorphic.

Lemma 1.1. *$\{\langle U_1, \cdots, U_k\rangle \mid U_1, \cdots, U_k$ open in $X\}$ forms a basis for $C(X)$.*

Proof. [6, Theorem 2.1].

Lemma 1.2. *$\{\pi_n'^{-1}(\langle U_1, \cdots, U_k\rangle) \mid n \in I$ and $\langle U_1, \cdots, U_k\rangle$ open in $C(X_n)\}$ forms a basis for $C_\infty(X)$.*

Proof. [1, Lemma 3.12, p. 218].

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LEMMA 1.3. $\{\langle \pi_n^{-1}(U_1), \cdots, \pi_n^{-1}(U_j)\rangle\}$ *forms a basis for* $C(X)$.

PROOF. If $V$ is a basic open set in $C(X)$, then $V=\langle V^1, \cdots, V^k\rangle = \{G \mid G\in C(X), G\subset \bigcup_{i=1}^k V^i \text{ and } G\cap V^i\neq\emptyset \text{ for } i=1, \cdots, k\}$. Let $e_0$ be the Lebesgue number of $(V^1, \cdots, V^k)$. Let $e_i>0$ for $i=1, \cdots, k$ be such that there exist $x^i\in V^i$ such that $S_{e_i}(x^i)\subset V^i$ where $S_{e_i}(x^i)$ is a spherical open set with center $x^i$ and radius $e_i$. Let $e=\min(e_i \mid i=0, 1, \cdots, k)$. Now there exist $n(e)$ and $\eta(e)$ such that if $A$ open subset of $X_n$ and diam $(A)<\eta$, then diam $\pi_n^{-1}(A)<e$. Cover $G_n=\pi_n(G)$ with open sets of diameter less than $\eta$, since $G_n$ is compact we need only a finite number of these open sets to cover $G_n$. Choose a finite irreducible set of such open sets and call them $T_1, \cdots, T_m$. We have $G_n\subset\bigcup_{j=1}^m T_j$ and $T_j\cap G_n\neq\emptyset$ for $j=1, \cdots, m$ and diam $(T_j)<\eta$.

So $G\subset\bigcup_{j=1}^m \pi_n^{-1}(T_j)$ and $G\cap\pi_n^{-1}(T_j)\neq\emptyset$ for $j=1, \cdots, m$ and diam $\pi_n^{-1}(T_j)<e$. Therefore $\pi_n^{-1}(T_j)$ is contained in some $V^i$. Let $T^i=\bigcup\{\pi_n^{-1}(T_j) \mid \pi_n^{-1}(T_j)\subset V^i\}$. Since for each $i$ we have $x_n^i=\pi_n(x^i)\in T_j$ and diam $(T_j)<\eta$ we have $x^i\in\pi_n^{-1}(T_j)$ and diam $\pi_n^{-1}(T_j)<e\leqq e_i$. Therefore there exists a $\pi_n^{-1}(T_j)\subset V^i$ for each $i$, so that $T^i\neq\emptyset$. Therefore $T^i$ is a nonnull open set of $X$ of the form $\pi_n^{-1}(\{\bigcup T_j \mid \pi_n^{-1}(T_j)\subset V^i\})$ where $\bigcup T_j$ is open in $X_n$.

Consider $\langle T^1, \cdots, T^k\rangle$ it is of the desired form. We must show $G\in\langle T^1, \cdots, T^k\rangle$ and $V$ is the union of such $\langle T^1, \cdots, T^k\rangle$'s.

First we show $G\in\langle T^1, \cdots, T^k\rangle$. $G\subset\bigcup_{j=1}^m \pi_n^{-1}(T_j)$ and $G\cap\pi_n^{-1}(T_j)\neq\emptyset$ for each $j$. Since $\bigcup_{j=1}^m \pi_n^{-1}(T_j)=\bigcup_{i=1}^k T^i$ we have $G\subset\bigcup_{i=1}^k T^i$ and $G\cap T^i\neq\emptyset$ for each $i$. Therefore $G\in\langle T^1, \cdots, T^k\rangle$.

Second we show $V=\bigcup\{\langle T^1, \cdots, T^j\rangle\}$. Since $G\in V$ implies $G\in\bigcup\{\langle T^1, \cdots, T^k\rangle\}$ we have $V\subset\bigcup\{\langle T^1, \cdots, T^k\rangle\}$.

If $A\in\bigcup\{\langle T^1, \cdots, T^j\rangle\}$ then $A\in\langle T^1, \cdots, T^j\rangle$ and so $A\subset\bigcup_{p=1}^j T^p$ and $A\cap T^p\neq\emptyset$ for each $p$. Therefore since $\bigcup_{p=1}^j T^p\subset\bigcup_{i=1}^k V^i$ we have $A\subset\bigcup_{i=1}^k V^i$ and $A\cap(\bigcup\{\pi_n^{-1}(T_m)\})\neq\emptyset$ where $\pi_n^{-1}(T_m)\subset V^i$ for $i=1, \cdots, k$. Therefore $A\subset\bigcup_{i=1}^k V^i$ and $A\cap V^i\neq\emptyset$ for $i=1, \cdots, k$. Therefore $A\in V$ and $\bigcup\{\langle T^1, \cdots, T^j\rangle\}\subset V$. Therefore

$$V=\bigcup\{\langle T^1, \cdots, T^j\rangle\}.$$

THEOREM 1.1. $C(X)$ *and* $C_\infty(X)$ *are homeomorphic.*

PROOF. If $A\in C_\infty(X)$ then $A=(A_1, A_2, A_3, \cdots)$ where $A_i\in C(X_i)$. If $D\in C(X)$ then $D=\{(x_1, x_2, \cdots) \mid x_i\in D_i=\pi_i(D)\}$. We define $h: C_\infty(X)\to C(X)$ by $h(A)=\{(x_1, x_2, \cdots) \mid x_i\in A_i\}$. If $h(A)=h(B)$ then $\{(x_1, x_2, \cdots) \mid x_i\in A_i\}=\{(y_1, y_2, \cdots) \mid y_i\in B_i\}$. Now for any $x_i\in A_i$, there is an $(x_1, x_2, \cdots, x_i, \cdots)$ equal to a $(y_1, \cdots, y_i, \cdots)$ and hence $x_i=y_i$ so that $x_i\in B_i$. Therefore $A_i\subset B_i$ and in the same way $B_i \subset A_i$ so that $A_i = B_i$ for each $i$. Therefore $A = B$ and $h$ is 1-1. If $B \in C(X)$ then $B = \{(x_1, x_2, \cdots) \mid x_i \in B_i\} = h((B_1, B_2, \cdots))$ so that $h$ is onto.

By Lemma 1.3 $\langle \pi_n^{-1}(U_1), \cdots, \pi_n^{-1}(U_k) \rangle$ is a basic open set so to show $h$ is continuous we will show that $h^{-1}(\langle \pi_n^{-1}(U_1), \cdots, \pi_n^{-1}(U_k) \rangle)$ is an open set in $C_\infty(X)$.

$$
\begin{aligned}
&h^{-1}(\langle \pi_n^{-1}(U_1), \cdots, \pi_n^{-1}(U_k) \rangle) \\
&\quad = \{ A \in C_\infty(X) \mid h(A) \in \langle \pi_n^{-1}(U_1), \cdots, \pi_n^{-1}(U_k) \rangle \} \\
&\quad = \left\{ A \in C_\infty(X) \mid h(A) \subset \bigcup_{i=1}^{k} \pi_n^{-1}(U_i) \text{ and } h(A) \cap \pi_n^{-1}(U_i) \neq \emptyset \right\} \\
&\quad = \left\{ A \in C_\infty(X) \mid \pi_n h(A) \subset \bigcup_{i=1}^{k} U_i \text{ and } \pi_n h(A) \cap U_i \neq \emptyset \right\} \\
&\quad = \{ A \in C_\infty(X) \mid \pi_n(\{(x_1, \cdots) \mid x_i \in A_i\}) \in \langle U_1, \cdots, U_k \rangle \} \\
&\quad = \{ A \in C_\infty(X) \mid \{x_n \mid x_n \in A_n\} \in \langle U_1, \cdots, U_k \rangle \} \\
&\quad = \{ A \in C_\infty(X) \mid A_n \in \langle U_1, \cdots, U_k \rangle \} \\
&\quad = \{ A \in C_\infty(X) \mid A \in \pi_n'^{-1}(\langle U_1, \cdots, U_k \rangle) \} \\
&\quad = \pi_n'^{-1}(\langle U_1, \cdots, U_k \rangle) \text{ open in } C_\infty(X).
\end{aligned}
$$

Property 3.2. *For $e > 0$, there exists $d(e) > 0$ such that if $a, b \in X$, $\operatorname{dist}(a, b) < d(e)$ and $a \in A \in C(X)$, then there exists $B$ such that $b \in B \in C(X)$ with the Hausdorff distance from $A$ to $B$ less than $e$.*

Theorem 1.2. *If $X$ is a metric continuum then $C(X)$ is acyclic in all dimensions.*

Proof. By [2, p. 183] $X = \lim (X_i, f_i, I)$ where $X_i$ is a polyhedron, $f_i$ is continuous and onto, $I$ is the set of natural numbers. If $Y$ has property 3.2 by [4, Theorem 3.4] the Vietoris groups $V_n(C(Y)) = 0$. Now a polyhedron $P$ has property 3.2 so $V_n(C(P)) = 0$. By [5, Theorem 26.1] for a compact metric space $Y$, $V_n(Y) = H_n(Y)$ where the Vietoris groups $V_n$ and the Čech groups $H_n$ are taken over a discrete group. So using the above, Theorem 1.1 and the continuity of Čech theory we have the following: $V_n(C(X)) = H_n(C(X)) = H_n(C_\infty(X)) = H_n(\lim (C(X_i), f_i', I)) = \lim (H_n(C(X_i)), f'_{i*}, I) = \lim (O_i, f'_{i*}, I) = 0$ where $O_i = 0$.

2. **Dimension of** $C(X)$. Kelley leaves as an open question the dimension of $C(X)$ when $X$ is not locally connected. If $X$ is a metric continuum of dimension $n$, then $X = \lim (X_i, f_i, I)$ where $X_i$ is a polyhedron of dimension $n$. If in addition $\dim C(X_i) \leq k$ for all $i$ we shall say $X$ has property k (with respect to $\lim (X_i, f_i, I)$).

THEOREM 2.1. *If* dim $(X)=1$ *and X has property* k, *then* dim $C(X) < \infty$.

PROOF. By [**4**, Theorem 5.4] (if $X$ is Peanian then dim $C(X) < \infty$ if and only if $X$ is a linear graph) we have since dim $C(X_i) \leqq k$ for all $i$ that $k < \infty$. Therefore dim $C(X) =$ dim $C_\infty(X) \leqq k < \infty$.

EXAMPLE 2.1. Let $X$ be the dyadic solenoid, then since $X = \lim (X_i, f_i, I)$ where $X_i = S^1$ and $f_i(z) = z^2$, we have dim $C(X_i) = 2$ for each $i$, hence the dim $C(X) =$ dim $C_\infty(X) \leqq 2$.

EXAMPLE 2.2. To see the need of imposing property k in Theorem 2.1 consider the following: let $X_i$ be the union of $2^i$ straight line segments $A_0^i, \cdots, A_{2^i-1}^i$ where $A_j^i$ $(j=0, \cdots, 2^i-1)$ is from $(0, 0)$ to $(1, j\pi/2^{i-1})$ in the plane (polar coordinates). Let $f_i \colon X_{i+1} \rightarrow X_i$ be the identity map on $A_0^{i+1}, A_2^{i+1}, A_4^{i+1}, \cdots, A_{2^i-1}^i$ where $f_i(A_j^{i+1}) = A_{j/2}^i$ for $j=0, 2, \cdots, 2^{i-1}$, and $f_i$ maps $A_j^{i+1}$ linearly onto $A_{(j-1)/2}^i$ keeping the origin fixed for $j=1, 3, \cdots, 2^i-1$. Then $X$ is a Cantor set of arcs meeting at a single point $(\bar{x}_i) = \bar{x}$ where $\bar{x}_i = (0, 0)$ for each $i$. Now the dim $C(X_i) = 2 + \sum_{\text{order}\, x_i \geqq 2}$ (order $x_i - 2) = 2 +$(order $\bar{x}_i - 2) =$ order $\bar{x}_i = 2^i$, so that $X$ fails to have property k. Further dim $C(X)$ is infinite since the order $\bar{x} = \infty$.

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