MEROMORPHIC FUNCTIONS OF FINITE ORDER

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1. Introduction. Let $f(z)$ be an entire function of finite order ρ. If the genus p of the canonical product (c.p.) formed with zeros of $f(z)$ is equal to the genus of $f(z)$, then it is known[1] that [**9**; **10**]

$$\liminf_{r\to\infty} \frac{\log M(r)}{n(r, 0)\phi(r)} = 0$$

where $\phi(r)$ is any positive nondecreasing function such that

$$\int_1^\infty dt/t\phi(t) \tag{1}$$

is convergent. Another proof of this theorem is given by Boas [**2**] where he assumes only that $\phi(r)$ is positive and (1) exists. Following the method of Boas, Shanker Hari has shown in a paper under publication that if $f(z)$ is a c.p. of order and genus ρ then

$$\liminf_{r\to\infty} \frac{\log M(r)}{N(r, 0)\phi(r)} = 0.$$

In this paper we prove a more general theorem by the method given in [**9**; **10**].

Let G, p_1, p_2 denote respectively the genus of a meromorphic function $f(z)$, the genus of the c.p. P_1 formed with zeros of f and the genus of the c.p. P_2 formed with poles of f. Write[2] $S=\max(p_1, p_2)$, $n(t) =n(t, f-0)+n(t, f-\infty)$, $n_1(t)=n(t, P_1-0)+n(t, P_2-0)$. Let k denote any constant $\geqq 1$, $\phi(x)$ any positive nondecreasing function such that (1) exists, $(a_i(z))$, $i=0, 1, \cdots, l$ any meromorphic functions (including rational functions or finite constants) such that

$$T(r, a_i(z)) = o(T(r, f)), \qquad i = 0, 1, \cdots, l \tag{2}$$

and $g_0(z)=g(z)$, $g_1(z)$, $g_2(z)$ any meromorphic functions (including rational functions or constants, finite or infinite) such that

$$T(r, g_i(z)) = o(T(r, f)), \qquad i = 0, 1, 2. \tag{3}$$

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[1] For notations see 7 [**7**; **8**].

[2] The function $n(r, f-g)$ denotes the number of zeros of $f-g$ in $|z|\leqq r$. If g is infinite constant, then $N(r, 1/(f-g))=N(r, f)$.

Write

$$\psi(f) = \sum_0^l a_i(z)f^{(i)}(z), \qquad \psi(f - g_1) = \sum_0^l a_i(z)\{f^{(i)}(z) - g_1^{(i)}(z)\}$$

when g_1 is not infinite constant; we suppose that (a_i) are so chosen that neither $\psi(f)$ nor $\psi(f-g_1)$ reduces identically to a constant.

THEOREM 1. *Let $f(z)$ be a meromorphic function of integer order ρ.*
(a) *If $G=S$, then*

$$\liminf_{r\to\infty} \frac{T(kr, f)}{\{n(r, 0) + n(r, \infty)\}\phi(r)} = 0, \tag{4}$$

$$\liminf_{r\to\infty} \frac{T(kr, f)}{\{N(r, 0) + N(r, \infty)\}\phi(r)} = 0. \tag{5}$$

(b) *If $G>S$, then*

$$\overline{N}\left(r, \frac{1}{\psi(f)-1}\right) \sim T(r, f) \sim Ar^\rho \qquad (0 < A < \infty). \tag{6}$$

THEOREM 2. *Let $f(z)$ be a meromorphic function of noninteger order $\rho>0$. Then*
(a)

$$\liminf_{r\to\infty} \frac{T(kr, f)}{n(r, f - g_1) + n(r, f - g_2)} \leqq k^{1+S}B(S), \tag{7}$$

$$\liminf_{r\to\infty} \frac{T(kr, f)}{N(r, 1/(f - g_1)) + N(r, 1/(f - g_2))} \leqq k^{1+S}(S + 1)B(S) \tag{8}$$

where

$$S = [\rho], \qquad A(S) = 2 \qquad \text{if } S = 0,$$
$$= 2e(2 + \log S)(S + 1)^2/S \qquad \text{if } S \geqq 1,$$

and

$$B(S) = A(S)/(\rho - S)(1 + S - \rho)$$

and g_1, g_2 are any two distinct functions satisfying (3).

COROLLARY.

$$\liminf_{r\to\infty} \frac{T(r, f)}{n(r, f - g)} \leqq 2B(S), \tag{9}$$

$$\liminf_{r\to\infty} \frac{T(r,f)}{N(r,1/(f-g))} \leqq 2(S+1)B(S) \tag{10}$$

except possibly for one g *in each relation* (9), (10).

(b) *If* (10) *does not hold for* $g=g_1$, *then for all* $r>r_0$,

$$T(r,f) < \left\{\frac{2(S+1)B(S)}{2(S+1)B(S)-1}\right\}\{N(r,f)+\overline{N}(r,1/(\psi(f-g_1)-1))\} \tag{11}$$

if $g_1 \not\equiv \infty$ *and*

$$T(r,f) < \left\{\frac{2(S+1)B(S)}{2(S+1)B(S)-1}\right\}\left\{N\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{\psi(f)-1}\right)\right\} \tag{12}$$

if $g_1 \equiv \infty$.

(c) *If* (9) *does not hold for* $g=g_1$, *then we have for a sequence of values of* $r\uparrow\infty$,

$$T(r,f) < \left\{\frac{2\rho B(S)}{2\rho B(S)-1}\right\}\{N(r,f)+\overline{N}(r,1/(\psi(f-g_1)-1))\} \tag{13}$$

if $g_1 \not\equiv \infty$ *and*

$$T(r,f) < \left\{\frac{2\rho B(S)}{2\rho B(S)-1}\right\}\{N(r,1/f)+\overline{N}(r,1/(\psi(f)-1))\} \tag{14}$$

if $g_1 \equiv \infty$.

THEOREM 3. *Let* $f(z)$ *be an entire function of order* ρ *and let, when* $\rho>0$, $h(\rho)=(\rho+(1+\rho^2)^{1/2})(1+(1+\rho^2)^{1/2})^\rho/\rho^\rho$. *If* $\rho>0$, *then*

$$\liminf_{r\to\infty} \frac{\log M(kr)}{n(r,f-g)} \leqq \frac{2k^\rho h(\rho)}{\rho}, \tag{15}$$

$$\liminf_{r\to\infty} \frac{\log M(kr)}{N(r,1/(f-g))} \leqq 2k^\rho h(\rho) \tag{16}$$

for every entire function $g(z)$ (*including a polynomial or a finite constant*) *satisfying* (3), *with one possible exception. If* $\rho=0$, *then*

$$\liminf_{r\to\infty} \frac{\log M(kr)}{N(r,1/(f-g))} = 1 \tag{17}$$

for every entire function $g(z)$ *such that* $M(r,g)=o(M(r,f))$.

THEOREM 4. *Let* $f(z)$ *be a meromorphic function of order* ρ. *If* $\rho>0$, *then*

$$\liminf_{r\to\infty} \frac{T(kr,f)}{n(r,f-g)} \leqq \frac{3k^\rho}{\rho}, \tag{18}$$

$$\liminf_{r\to\infty} \frac{T(kr,f)}{N(r,1/(f-g))} \leqq 3k^\rho \tag{19}$$

for every meromorphic function $g(z)$ *satisfying* (3), *with two possible exceptions. If* $\rho=0$ *then*

$$\liminf_{r\to\infty} \frac{T(kr,f)}{N(r,1/(f-g_1))+N(r,1/(f-g_2))} \leqq 1 \tag{20}$$

where g_1, g_2 *are any two distinct functions satisfying* (3).

2. **Remarks.** (i) Let $\delta(\alpha)$ denote Nevanlinna defect. If $\sum\delta(\alpha)<2$, then $G=S$ and (4) holds. If $\sum\delta(\alpha)=2$, then (4) may or may not hold. (See [**13**, pp. 590–591] for an entire function of which the genus is greater than the genus of its c.p. and for which $\sum\delta(\alpha)=2$ and (4) holds.) Conversely if (4) holds, then $\sum\delta(\alpha)$ may have its maximum value 2 or $\sum\delta(\alpha)<2$ ($f(z)=\sin z$). If (4) does not hold then $G>S$, $\delta(0)=\delta(\infty)=1$.

(ii) For entire functions of order ρ, $0\leqq\rho<1$, it is known that [**1**; **11**]

$$\liminf_{r\to\infty} \frac{\log M(r)}{N(r,a)} \leqq \frac{1}{1-\rho}$$

and for meromorphic functions of order ρ, $0\leqq\rho<1$, it is known that

$$\liminf_{r\to\infty} \frac{T(r)}{N(r,a)+N(r,b)} \leqq \frac{1}{1-\rho}.$$

More precise results in this direction have been given by Edrei and Fuchs [**4**].

(iii) For entire functions of noninteger order $\rho>0$, it is known that

$$\mathrm{L}(f,a) = \liminf_{r\to\infty} \frac{\log M(r)}{n(r,f-a)} < \infty$$

for every finite a. A sharp upper bound for $L(f,a)$ when $\rho>1$, is not known but by (15) $L(f,a)\leqq 2h(\rho)/\rho$ except possibly for one value of a. This exceptional value may exist. Consider [**7**, pp. 18–19]

$$f(z) = \prod_2^\infty E\left(\frac{z}{a_n}, p\right), \qquad a_n = -n^{1/\rho}, \qquad p<\rho<p+1.$$

$f(z)$ is an entire function of order ρ and

$$\log M(r)/n(r, 0) \sim \pi/\operatorname{Sin} \pi(\rho - p)$$

which can be made greater than $2h(\rho)/\rho$ by choosing $\rho - p$ sufficiently small.

(iv) If (15), (16), (18) or (19) do not hold for $g = g_1$ then we can obtain relations similar to (11)–(14). For instance if (15) or (16) does not hold when $g = g_1$, then

$$\liminf_{r\to\infty} \frac{T(r, f)}{\overline{N}(r, 1/(\psi(f - g_1) - 1))} \leqq 2.$$

3. Theorem 1.

LEMMA 1. *If $f(z) = z^v \exp(Q(z))P_1(z)/P_2(z)$ is a proper meromorphic function*[3] *of order $\rho \geqq 0$ such that $G = S$, then for all $r > r_0$*

$$T(kr, f) \leqq A(S)k^{1+S}J(r, S)$$

where[4]

$$\begin{aligned} A(S) &= 2 && \text{if } S = 0, \\ &= 2e(2 + \log S)(S + 1)^2/S && \text{if } S \geqq 1, \end{aligned}$$

and

$$J(r, S) = r^{1+S} \int_0^\infty \frac{n_1(t)dt}{t^{1+S}(t + r)}.$$

PROOF. We have

$$T(kr, f) = O(r^S + \log r) + \log M(kr, P_1) + \log M(kr, P_2)$$

and when $p_1 \geqq 1$ [8, pp. 225–226]

$\log M(kr, P_1)$

$$\leqq (2 + 1/p_1)e(2 + \log p_1)(1 + p_1)k^{1+p_1}r^{1+p_1} \int_0^\infty \frac{n(t, P_1 - 0)dt}{t^{1+p_1}(t + kr)}.$$

Hence, when $p_1 = p_2 \geqq 1$,

$$T(kr, f) \leqq \frac{2e(2 + \log S)(1 + S)^2}{S} k^{1+S}J(r, S).$$

If $p_1 = p_2 = 0$, then

$$T(kr, f) < O(\log r) + kr \int_0^\infty \frac{n_1(t)dt}{t(t + kr)}.$$

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[3] f is not a rational function.

[4] We can take $A(0) = 1 + c$, where c is any positive number.

If $p_1 > p_2$ then $\log M(kr, P_2) = o(r^{1+p_2}) = o(r^{p_1})$ and a similar result if $p_1 < p_2$. Hence the lemma is proved.

LEMMA 2. *If $f(z)$ is a proper meromorphic function of finite order and*

$$\psi(f) = \psi(z) = \sum_0^l a_i(z) f^{(i)}(z)$$

then

$$(21) \quad (1 + o(1))T(r,f) < \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{\psi(f) - 1}\right) + N\left(r, \frac{1}{f}\right).$$

This lemma is substantially contained in Theorem 7 of [5].

LEMMA 3. *If $f(z) = z^v \exp(Q(z)) P_1(z)/P_2(z)$ and $G > S$ then $Q(z) = a_\rho z^\rho + \cdots$, $|a_\rho| = A \neq 0$ and*

(i) $T(r, f) \sim A r^\rho$,

(ii) $T(r, f') \sim A r^\rho$,

(iii) $T(r, \psi) = T(r, \sum_0' a_i f^{(i)}) \sim A r^\rho$.

PROOF. Write $f = b_0 \exp(Q)$ where $b_0 = z^v P_1 P_2^{-1}$,

$$T(r, f) \leqq T(r, \exp Q) + T(r, b_0),$$
$$T(r, \exp Q) \leqq T(r, f) + T(r, b_0) + O(1)$$

and since $T(r, \exp Q) \sim A r^\rho$, $T(r, b_0) = o(r^\rho)$, (i) follows.

$$\text{(ii)} \quad \begin{aligned} f' &= e^Q \{ k z^{k-1} P_1 P_2^{-1} + z^k Q' P_1 P_2^{-1} + z^k P_1' P_2^{-1} - Z^k P_1 P_2^{-2} P_2' \} \\ &= e^Q b_1 \text{ (say)}. \end{aligned}$$

Then $T(r, b_1) = o(r^\rho)$ and so $T(r, f') \sim A r^\rho$.

(iii) $f'' = (\exp Q) b_2$ where $T(r, b_2) = o(r^\rho)$ and so on. Hence

$$\begin{aligned} T(r, \psi(f)) &= T(r, (\exp Q) \textstyle\sum a_i b_i) \\ &\leqq T(r, \exp Q) + T(r, \textstyle\sum a_i b_i) \\ &\leqq (A + o(1)) r^\rho + \textstyle\sum T(r, a_i) + \sum T(r, b_i) \\ &= (A + o(1)) r^\rho. \end{aligned}$$

Also

$$\exp Q = \psi(f) / \textstyle\sum a_i b_i, \qquad T(r, \exp Q) \leqq T(r, \psi(f)) + o(r^\rho)$$

and hence $T(r, \psi(f)) \sim A r^\rho$.

(a) PROOF OF[5] (4). Since $\rho \geqq 1$ and $G = S$, $n(x)$ tends to infinity with

[5] (4) can also be proved by the method given in [12, pp. 696–697].

x. Further $T(kr, f) \sim T(kr, fz^v)$ where v is any integer and so we may suppose $f(0) \neq 0, \neq \infty$. Then $f(z) = \exp(Q(z))P_1(z)/P_2(z)$. Since $G = S$, we have $S = \rho$ or $\rho - 1$. Consider first when $S = \rho$. If $n(x) = O(x^S)$ we have

$$J(r, S) < c_1 r^S \int_0^r \frac{n(x)dx}{x^{1+S}}$$

and hence from Lemma 1 of [**9**, p. 24] we get

$$\liminf_{r \to \infty} J(r, S)/n(r)\phi(r) = 0.$$

If $\limsup_{x \to \infty} n(x)/x^S = \infty$, we write $y = \log x$ and $\log(n(x)/x^S) = \log \mu(x) = \psi(y)$. Then $\limsup_{y \to \infty} \psi(y) = \infty$, and since $f(z)$ is of order $\rho = S$, $\limsup_{y \to \infty} \psi(y)/y = 0$. Apply Lemma 2 of [**9**, p. 25]. There exists a sequence $r_n \uparrow \infty$ for which

$$\mu(x) < \mu(r_n), \qquad \Delta < x < r_n,$$

$$\frac{\log \mu(x)}{\log x} < \frac{\log \mu(r_n)}{\log r_n}, \qquad x > r_n.$$

Hence

$$J(r_n, S) < c_2 n(r_n) \log r_n.$$

Consider now $S = \rho - 1$. Given $\phi(x)$ we can find [**10**] a function $\phi_1(x)$ such that for all large x: (i) $\phi_1(x) \leqq \phi(x)$; (ii) $\phi_1(x)/x^\alpha$ is nonincreasing, α being a positive number less than unity; (iii) $\int_1^\infty dt/t\phi_1(t)$ is convergent. If now

$$\limsup_{x \to \infty} n(x)\phi_1(x)/x^{1+S} > 0,$$

then for a sequence $R_n \uparrow \infty$, $J(R_n, S) = o(n(R_n)\phi_1(R_n))$. If $n(x)\phi_1(x) = o(x^{1+S})$ then we choose in Lemma 3 of [**10**, p. 184] $0 < \beta < 1 - \alpha$, $\theta(x) = x^\beta$, $\Psi(x) = n(x)\phi_1(x)/x^{S+1-\beta}$ and obtain for $x_n \uparrow \infty$, $J(x_n, S) = o(\phi(x_n)n(x_n))$ and (2) is proved.

Since $n(r, a) \leqq N(2r, a)/\log 2$, (5) follows from (4).

(b) Proof of (6). By hypothesis

$$N(r, f) = o(r^\rho), \qquad N(r, 1/f) = o(r^\rho),$$

and so from (21)

$$(1 + o(1))T(r, f) < \overline{N}(r, 1/(\psi(f) - 1)) < T(r, \psi(f)) + O(1)$$

and so (6) follows from Lemma 3(i) and (iii).

4. Theorems 2, 3, 4.

LEMMA 4. *If $f(z)$ is a meromorphic function of order $\rho>0$, then*

$$I_1(r) = \int_1^r \frac{T(kt,f)}{t}\,dt < \frac{k^\rho}{\rho}(1+\epsilon)T(r,f)$$

for a sequence of values of r tending to infinity.

LEMMA 5. *If $f(z)$ is an entire function of order $\rho>0$, then*

$$I_2(r) = \int_1^r \frac{\log M(kt)}{t}\,dt < \frac{k^\rho}{\rho}h(\rho)(1+\epsilon)T(r,f)$$

where $h(\rho)$ has been defined in the statement of Theorem 3, for a sequence of values of r tending to infinity.

The proof of Lemma 4 is straightforward (cf. [**14**, p. 321]) and the proof of Lemma 5 depends on the relation

$$\log M(r) \leqq \frac{h+1}{h-1}T(hr), \qquad h>1.$$

We obtain for a sequence of values of $r\to\infty$,

$$I_2(r) \leqq \frac{h+1}{h-1}\,\frac{h^\rho k^\rho}{\rho}(1+\epsilon)T(r).$$

Choose $h=(1+(1+\rho^2)^{1/2})/\rho$.

(a) PROOF OF (7). We have $S=[\rho]$

$$T(kr,f) < A(S)k^{1+S}J(r,S), \qquad r>r_0.$$

We choose in Lemma 3 of [**10**, p. 184] (see also [**12**, p. 69]) $\Psi(x)=n_1(x)/x^{\rho-\epsilon}$ and obtain that for $r=r_n\uparrow\infty$

$$T(kr_n,f) < B(S)(1+\epsilon')k^{1+S}n(r_n).$$

Let $H(z)=(f(z)-g_1(z))/(f(z)-g_2(z))$, $g_1\not\equiv\infty$, $g_2\not\equiv\infty$. Then $T(r,H)\sim T(r,f)$. Also for a sequence $R_n\uparrow\infty$,

$$T(kR_n,H) < B(S)(1+\epsilon')k^{1+S}\{n(R_n,H-0)+n(R_n,H-\infty)\}.$$

Now

$$n(R_n,H-0)+n(R_n,H-\infty) < \sum_1^2 n(R_n,f-g_i)+\sum_1^2 n(R_n,g_i-\infty).$$

Suppose now $k>1$. Then

$$n(R_n,g_i-\infty) < N(kR_n,g_i)/\log k = o(T(kR_n,f)).$$

Hence for $n > n_1$

$$T(kR_n, f) < B(S)(1+\epsilon')k^{1+S} \sum_1^2 n(R_n, f - g_i). \tag{22}$$

If we take $H(z) = f(z) - g_2(z)$, $g_2 \not\equiv \infty$, we get for another $\{R_n\} \uparrow \infty$,

$$T(kR_n, f) < B(S)(1+\epsilon')k^{1+S}\{n(R_n, f-\infty) + n(R_n, f - g_2)\} \tag{23}$$

and so from (22) and (23)

$$\liminf_{r\to\infty} T(kr, f) \Big/ \sum_1^2 n(r, f - g_i) \leqq B(S)k^{1+S} \tag{24}$$

and (7) is proved when $k > 1$. Further k can be chosen arbitrarily near to 1 in the relation

$$\liminf_{r\to\infty} T(r, f) \Big/ \sum_1^2 n(r, f - g_i) \leqq B(S)k^{1+S}$$

and so (7) is true when $k \geqq 1$.

Proof of (8). Write

$$N_1(t) = \int_0^r \{n_1(t)/t\}\,dt; \qquad N(t) = N(t, f) + N(t, 1/f).$$

Then

$$\begin{aligned} J(r, S) &= \int_0^\infty \frac{r^{1+S} n_1(t)\,dt}{t^{1+S}(t+r)} \\ &< (S+1)\left\{r^S \int_0^r \frac{N_1(t)\,dt}{t^{1+S}} + r^{1+S}\int_r^\infty \frac{N_1(t)\,dt}{t^{2+S}}\right\} \\ &< (S+1)(1+\epsilon)\left\{r^S \int_{x_0}^r \frac{[N(t)]\,dt}{t^{1+S}} + r^{1+S}\int_r^\infty \frac{[N(t)]\,dt}{t^{S+2}}\right\} \end{aligned}$$

if $r > x_1(\epsilon)$. Hence by Lemma 3 of [**10**, p. 184] we get

$$\liminf_{r\to\infty} \frac{T(kr, f)}{N(r)} \leqq B(S)(1+S)k^{1+S}$$

and the rest of the argument is the same as for (7). Hence (8) is proved. The Corollary follows from (7) and (8).

(b) If (10) is false for $g = g_1 \not\equiv \infty$, then

$$\liminf_{r\to\infty} \frac{T(r, f)}{N(r, 1/(f - g_1))} = \beta > 2(S+1)B(S).$$

Hence for all $r > r_0(\epsilon)$,

$$T(r, f) > (\beta - \epsilon)N(r, 1/(f - g_1))$$

We apply Lemma 2 to $f - g_1$ and obtain

$$\{1 + o(1)\}T(r, f - g_1)$$
$$\leqq \overline{N}(r, f - g_1) + \overline{N}\left(r, \frac{1}{\psi(f - g_1) - 1}\right) + N\left(r, \frac{1}{f - g_1}\right),$$

and for $r > r_1(\epsilon)$

$$\left(1 - \frac{1}{\beta - \epsilon} - \epsilon\right) T(r, f) \leqq N(r, f) + \overline{N}\left(r, \frac{1}{\psi(f - g_1) - 1}\right).$$

Hence for $r > r_2$

$$T(r, f) < \left\{\frac{2(S + 1)B(S)}{2(S + 1)B(S) - 1}\right\} \left\{N(r, f) + \overline{N}\left(r, \frac{1}{\psi(f - g_1) - 1}\right)\right\}.$$

The proof of (12) is similar.

(c) We omit the proofs of (13)–(14) which can be proved with the help of Lemma 4.

(d) The first part of Theorem 3 follows from Lemma 5 and (17) follows from a result of Boas [**1**, pp. 6–7; **3**, p. 48]. We omit the proofs of Theorems 3 and 4.

5. **Example.** We show that (4), (5) are best possible in the sense that k cannot be replaced by a function $\alpha(r)$ tending to infinity with r. We prove that *given* $\alpha(r) \to \infty$ *with* r, *there is a function* $\phi(x)$ *such that* (1) *exists, and an entire function* (*c.p.*) $f(z)$ *of integer order* ρ *and genus* $\rho - 1$ *such that*

$$(25) \quad \liminf_{r\to\infty} \frac{T(r\alpha(r), f)}{N(r, 0)\phi(r)} = \liminf_{r\to\infty} \frac{T(r\alpha(r), f)}{n(r, 0)\phi(r)} \geqq \frac{1}{(1 + 2^{1/2})^2} \quad \text{if } \rho = 1$$
$$= \infty \quad \text{if } \rho > 1.$$

(a) We suppose that $\alpha(r)$ satisfies the following:

(i) $\alpha(r) > 0$ for $r \geqq x_0 > 0$ and tends to infinity with r.

(ii) $\alpha'(r) > 0$ and $r\alpha'(r) \downarrow$ for $r \geqq x_0$.

(iii) $r^2\alpha'(r)/\alpha^2(r)$ is strictly increasing for $r \geqq x_0$. For instance we can take $\alpha(r) = l_k r$, $k = 1, 2, \cdots$, ($l_1 r = \log r$; $l_k(r) = \log(l_{k-1}r)$). A function of slower growth and satisfying (i)–(iii) is given by the functional equation $\alpha(e^x) = e\alpha(x)$. Let

$$\phi(x) = \alpha^2(x)/x\alpha'(x) \qquad x \geqq x_0,$$
$$= \phi(x_0) \qquad 0 \leqq x < x_0,$$

$$n(x, 0) = [x/\phi(x)] \qquad x \geqq x_0,$$
$$= 0 \qquad 0 \leqq x < x_0.$$

From (ii), (iii) it is seen that $\phi(x)$ and $x/\phi(x) \uparrow$ with x. Further

$$\int_0^X \frac{n(t, 0)}{t^2} dt < \int_{x_0}^X \frac{dt}{t\phi(t)} \to \frac{1}{\alpha(x_0)} \tag{26}$$

as $X \to \infty$. Hence (1) is convergent. Write $R = r\alpha(r)$. Then

$$\alpha(R) - \alpha(r) = \int_r^R \alpha'(t)dt = o(\alpha(R)).$$

Hence $\alpha(R) \sim \alpha(r)$ and so $\alpha(r)$ is "slowly increasing" function [6] and

$$\alpha(r) = o(r^\delta), \qquad \delta > 0, \qquad \int_{x_0}^r \frac{dt}{\alpha(t)} \sim \frac{r}{\alpha(r)}.$$

From (ii) we have $\alpha(r) = o(\phi(r))$. Let

$$f(z) = \prod_1^\infty \left(1 + \frac{z}{r_n}\right)$$

where $r_n > 0$, $n = 1, 2, \cdots$, and are given by $n(x, 0)$. (The first few zeros may be equal; cf. [8, p. 228].) From (26) we see that the series $\sum 1/r_n$ is convergent and $f(z)$ is an entire function (c.p.) of genus $p = 0$. Now

$$\log M(r) < r\int_0^r \frac{n(t, 0)}{t} dt + r\int_r^\infty \frac{n(t, 0)}{t^2} dt \leqq \{1 + o(1)\} \frac{r}{\alpha(r)},$$

$$\log M(r) > r\int_R^\infty \frac{n(t, 0)dt}{t(t + r)}$$
$$> r\int_R^\infty \left\{\frac{t}{\phi(t)} - 1\right\} \frac{dt}{t(t + r)} = \frac{r}{\alpha(r)} \{1 + o(1)\}.$$

Hence

$$\log M(r) \sim r/\alpha(r), \log M(R) \sim r, n(r, 0)\phi(r) \sim r \sim N(r, 0)\phi(r),$$

$$T(R) \geqq \frac{\beta - 1}{\beta + 1} \log M\left(\frac{R}{\beta}\right), \qquad \beta > 1,$$
$$\sim \frac{\beta - 1}{\beta + 1} \frac{R}{\beta\alpha(r)} = \frac{\beta - 1}{\beta + 1} \frac{r}{\beta}. \tag{27}$$

Choose $\beta = 1 + 2^{1/2}$ and (25) follows when $\rho = 1$.

(b) If $\rho > 1$ then let

$$f(z) = \prod_1^\infty \left\{1 + \frac{z^\rho}{r_n}\right\}$$

where r_n have been defined in (a). $f(z)$ is an entire function (c.p.) of order ρ and genus $\rho - 1$. We have for all $r > r_0$

$$\log M(kr, f) < c_3 n(r, f - 0)\phi(r)/\alpha(r),$$

$$\log M(R, f) \geqq \frac{1}{\rho}(1 + o(1))(\alpha(r)^{\rho-1} n(r, f - 0)\phi(r),$$

$$T(R, f) > c_4(\alpha(r))^{\rho-1} n(r, f - 0)\phi(r),$$

$$T(R, f) > c_4(\alpha(r))^{\rho-1} N(r, 1/f)\phi(r)$$

and (25) is proved for $\rho > 1$.

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