

for suitable  $\psi$  and  $p$ . Since  $F$  is arbitrary in  $\mathfrak{F}(X_1)$ , the relation for  $s(g)(t)$  follows. This conclusion can also be deduced from a theorem of Milgram [2, p. 383].

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## SIMULTANEOUS APPROXIMATION BY A POLYNOMIAL AND ITS DERIVATIVES

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1. **Introduction.** We shall prove the following theorem:

**THEOREM.** *If  $f_0$  is a continuous function on a compact set  $C$  without interior and with connected complement of the complex plane, and if  $f_1, \dots, f_n$  are continuous functions on a compact totally disconnected subset  $E$  of  $C$ , then there exists a sequence  $\{p_i\}$  of polynomials such that  $p_i \rightarrow f_0$  uniformly on  $C$  as  $i \rightarrow \infty$ , and  $p_i^{[k]} \rightarrow f_k$  uniformly on  $E$  as  $i \rightarrow \infty$ , for  $1 \leq k \leq n$ , where  $p_i^{[k]}$  is the  $k$ th derivative of  $p_i$ .*

For the case in which  $E$  is void, this reduces to a well-known theorem of Lavrentiev [3]. A different case of the problem of simultaneous approximation by a polynomial and its derivatives—the problem of simultaneous approximation on a Jordan arc—has been considered in a previous paper [1], by methods distinct from the methods of the present paper.

We make some preliminary remarks concerning distributions. The distributions in question will all be defined on the complex plane and have complex values and have a compact support. If  $T$  is such a distribution and if  $f$  is a function defined on the complex plane and having partial derivatives of all orders, then  $\langle f, T \rangle$  will denote the

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value of  $T$  on  $f$ . Thus if  $T$  is a measure  $\mu$  we have  $\langle f, \mu \rangle = \int f d\mu$ , and if  $T$  is an integrable function  $h$  we have  $\langle f, h \rangle = \int f(z)h(z)dx dy$ . By Schwartz [4, p. 70], we have  $(\partial/\partial z^*)[(\pi z)^{-1} * T] = T$ .

**2. Proof of the Theorem.** Let  $\mu_0$  be any measure on  $C$ , and let  $\mu_1, \dots, \mu_n$  be measures on  $E$  such that  $\int p d\mu_0 + \int p^{(1)} d\mu_1 + \dots + \int p^{(n)} d\mu_n = 0$  for all polynomials  $p$ , where by measure we mean finite, complex-valued, Borel measure. If we let  $T$  be the distribution  $T = \sum_{k=1}^n (-1)^k [\partial/\partial z]^k \mu_k$ , it follows that the support of  $T$  is a subset of  $E$  and that  $\langle p, T + \mu_0 \rangle = 0$  for all polynomials  $p$ . Define the distribution  $R = (\pi z)^{-1} * (T + \mu_0)$ , so that  $(\partial/\partial z^*)(R) = T + \mu_0$ . Also,  $R$  agrees on the complement of  $C$  with the analytic function  $\phi(z) = \langle [\pi(z-\zeta)]^{-1}, (T + \mu_0)(\zeta) \rangle$  of  $z$ . Since the complement of  $C$  is connected, for each  $z$  in the complement of  $C$  the function  $(z-\zeta)^{-1}$  of  $\zeta$  can be uniformly approximated on some neighborhood of  $C$  by polynomials, so that the derivatives of  $(z-\zeta)^{-1}$  will be uniformly approximated on some neighborhood of  $C$  by the corresponding derivatives of the approximating polynomials. Since  $T + \mu_0$  is orthogonal to all polynomials, we see that  $\phi$  therefore vanishes on the complement of  $C$ . Thus the support of  $R$  is a subset of  $C$ . Now  $(\pi z)^{-1}$  is clearly the sum of an integrable function  $\phi_1$  and a continuous function  $\phi_2$ . Since the measure  $\mu_0$  has compact support,  $\phi_1 * \mu_0$  is an integrable function and  $\phi_2 * \mu_0$  is a continuous function. (See [4].) Therefore  $(\pi z)^{-1} * \mu_0$  is a locally integrable function. Since also  $(\pi z)^{-1} * T$  agrees with an analytic function on the complement of  $E$ , the distribution  $R = (\pi z)^{-1} * \mu_0 + (\pi z)^{-1} * T$  will agree with a locally integrable function  $h$  on the complement of  $E$ . Since  $R$  vanishes on the complement of  $C$ ,  $h$  will vanish on the complement of  $C$ .

Now let  $\zeta$  be any point in  $C - E$ , and let  $f$  be an infinitely differentiable function with compact support, such that  $f(z) = 1$  for all  $z$  in some neighborhood  $U$  of  $\zeta$  and such that the support  $H$  of  $f$  does not intersect  $E$ . Then the distribution  $(\partial f/\partial z^*)R$  is equal to the integrable function  $(\partial f/\partial z^*)h = g$ , so that  $\partial(fR)/\partial z^* = (\partial f/\partial z^*)R + f\partial R/\partial z^* = g + f(T + \mu_0) = g + f\mu_0$ , since  $H$  is disjoint from  $E$ . Since the support of  $h$  is a subset of  $C$  and the support of  $\partial f/\partial z^*$  is a subset of  $H - U$ , the support of  $g$  is a subset of  $H \cap C - U$ . Thus the distribution  $g + f\mu_0$  is a measure  $\nu$  on  $H \cap C$  which agrees with  $\mu_0$  on all subsets of  $U$ . For any polynomial  $p$ , we have  $\langle p, \nu \rangle = \langle p, \partial(fR)/\partial z^* \rangle = -\langle \partial p/\partial z^*, fR \rangle = -\langle 0, fR \rangle = 0$ . Thus  $\nu$  is orthogonal to all polynomials.

By the hypothesis of the theorem,  $H \cap C$  will have no interior and will have connected complement. By the theorem of Lavrentiev, any

continuous function on  $H \cap C$  can be uniformly approximated by polynomials. Thus  $\nu$  is orthogonal to any continuous function on  $H \cap C$ , so that  $\nu = 0$ . It follows that  $\mu_0$  vanishes on all subsets of the neighborhood  $U$  of  $\zeta$ . Since  $\zeta$  was an arbitrary point in  $C - E$ ,  $\mu_0$  is therefore a measure on  $E$ .

If  $D$  is any open and closed subset of  $E$ , there will exist, by Runge's theorem, a sequence  $\{p_i\}$  of polynomials converging uniformly to 1 on some neighborhood of  $D$  and to 0 on some neighborhood of  $E - D$ . Thus all derivatives  $p_i^{[k]}$  will converge uniformly to 0 on  $E$  as  $i \rightarrow \infty$ . Thus we see that  $0 = \int p_i d\mu_0 + \int p_i^{[1]} d\mu_1 + \cdots + \int p_i^{[n]} d\mu_n \rightarrow \mu_0(D)$  as  $i \rightarrow \infty$ , so that  $\mu_0$  vanishes on all open and closed subsets of  $E$ . Since  $E$  is compact and totally disconnected, the open and closed subsets of  $E$  form a basis for the open sets of  $E$  (see [2, p. 20]). Therefore  $\mu_0 = 0$ . Thus  $\int p d\mu_1 + \int p^{[1]} d\mu_2 + \cdots + \int p^{[n-1]} d\mu_n = 0$  for all polynomials  $p$ . By repeating the above argument, we can therefore show successively that  $\mu_0 = \mu_1 = \cdots = \mu_n = 0$ .

Let  $V$  denote the Banach space consisting of all  $n+1$ -tuples  $(f_0, f_1, \cdots, f_n)$ , where  $f_0$  is a continuous function on  $C$  and  $f_k$  for  $1 \leq k \leq n$  is a continuous function on  $E$ , under the norm  $\|(f_0, \cdots, f_n)\| = \|f_0\| + \cdots + \|f_n\|$ , where  $\|f_k\|$  is defined in the usual fashion. The dual  $V^*$  of  $V$  then consists of all  $n+1$ -tuples of measures  $(\mu_0, \mu_1, \cdots, \mu_n)$ , where  $\mu_0$  is a measure on  $C$  and  $\mu_k$  for  $1 \leq k \leq n$  is a measure on  $E$ . We have just shown that the only element of  $V^*$  which vanishes on all elements of  $V$  of the form  $(p, p^{[1]}, \cdots, p^{[n]})$ , where  $p$  is a polynomial, is the zero element. From the Hahn-Banach theorem it follows that any element in  $V$  is a limit of elements of the form  $(p, p^{[1]}, \cdots, p^{[n]})$ , and this is just the theorem.

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