1.1. Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let the sequence $\{t_n\}$ be defined by

$$t_n = \frac{(n+1)^{-1}s_0 + n^{-1}s_1 + \cdots + 1 \cdot s_n}{P_n},$$

$$P_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n+1}.$$

The series $\sum a_n$ is defined to be summable by harmonic means if the sequence $\{t_n\}$ tends to a limit as $n \to \infty$ [4]. If the series $\sum |t_n - t_{n-1}|$ is convergent, we say that the series is absolutely harmonic summable. It is known that the method of summability is absolutely regular and implies absolute Cesàro summability of every positive order [2].

1.2. Let $f(t)$ be a periodic function, with period $2\pi$, and integrable $(L)$ over $(-\pi, \pi)$. We assume without any loss of generality that the Fourier series of $f(t)$ is given by

$$\sum_{n=0}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t),$$

and that $\int_{-\pi}^{\pi} f(t) dt = 0$. We write

$$\phi(t) = \frac{1}{2} \{f(x + t) + f(x - t)\}.$$

Mohanty [3] has considered the absolute Riesz summability of the series

$$\sum_{n=1}^{\infty} A_n(t)/\log (n + 1).$$

2. In this paper we establish the following theorem:

**Theorem.** If $\phi(t)$ is of bounded variation in $(0, \pi)$ then the series (1.22) is absolutely summable by harmonic means.

We require the following lemmas for the proof of our theorem:

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Lemma 1 [5]. Uniformly for $0 < t < \pi$

$$\left| \sum_{m}^{n} \frac{\sin vt}{\nu} \right| \leq K$$

where $m$ and $n$ are any positive integers.

Lemma 2 [1]. If $0 < t < \pi$, then

$$\left| \sum_{k=0}^{m} \frac{\cos(k + 1)t}{k + 1} \right| = O\left(1 + \log \frac{1}{t}\right).$$

With the help of Lemmas 1 and 2, we may easily deduce

Lemma 3. If $0 < t < \pi$, then for all positive integers $m$ and $m'$

$$\sum_{k=m}^{m'} \frac{\sin(n - k)t}{n + 1} = O\left(1 + \log \frac{1}{t}\right).$$

Lemma 4. If $P_n = 1 + 1/2 + \cdots + 1/(n+1)$, then

(i) \begin{align*}
\sum_{k=0}^{[n/2]-2} & \Delta \left( \frac{P_n(n + 1) - P_k(k + 1)}{(n - k) \log(n - k + 1)} \right) = O(1); \\
\sum_{k=[n/2]}^{n-2} & \Delta \left( \frac{P_k \cdot \frac{1}{k + 1}}{\log(n - k + 1)} \right) = O\left(\frac{P_n}{n}\right).
\end{align*}

For proving (i) we observe that

\begin{align*}
\sum_{k=0}^{[n/2]-2} & \Delta \left( \frac{P_n(n + 1) - P_k(k + 1)}{(n - k) \log(n - k + 1)} \right) \\
\leq & \sum_{k=0}^{[n/2]-2} \frac{P_n(n + 1) - P_k(k + 1)}{(n - k)(n - k - 1) \log(n - k + 1)} \\
& + \sum_{k=0}^{[n/2]-2} \frac{P_{k+1}(k + 2) - P_k(k + 1)}{(n - k) \log(n - k + 1)} \\
& + O\left( \sum_{k=0}^{[n/2]-2} \frac{P_n(n + 1) - P_k(k + 1)}{(n - k)^2 \log^2(n - k + 1)} \right) \\
= & O\left( \sum_{k=0}^{[n/2]-2} \frac{1}{n - k} \right) + O\left[ \sum_{k=0}^{[n/2]-2} \frac{P_{k+1}}{(n - k) \log(n - k + 1)} \right] \\
& + O\left[ \sum_{k=0}^{[n/2]-2} \frac{1}{(n - k) \log(n - k)} \right] \\
= & O\left( \sum_{k=0}^{[n/2]-2} \frac{1}{(n - k)} \right) = O(1).
\end{align*}
Again
\[
\sum_{k=[n/2]}^{n-2} \left| \Delta \left( \frac{P_k}{k+1} \cdot \frac{1}{\log(n - k + 1)} \right) \right|
\]
\[= O\left( \sum_{k=[n/2]}^{n-2} \frac{P_k}{k+1} \frac{1}{(n-k) \log^2(n - k + 1)} \right) \]
\[+ O\left( \sum_{k=[n/2]}^{n-2} \frac{P_k}{(k+1)^2 \log(n - k + 1)} \right) \]
\[= O(P_n/n) \left( \sum_{k=[n/2]}^{n-2} \frac{1}{(n-k) \log^2(n - k + 1)} \right) + O\left( \frac{P_n}{n^2} \right). (n) \]
\[= O(P_n/n). \]
This proves the lemma completely.

3. Proof of the theorem. Since
\[l_n = \frac{P_n u_0 + P_{n-1} u_1 + \cdots + P_0 u_n}{P_n}, \quad \left( u_n = \frac{A_n(t)}{\log(n + 1)} \right), \]
we have
\[l_n - l_{n-1} = \sum_{\nu=0}^{n-1} \frac{P_\nu}{P_n} - \frac{P_{\nu-1}}{P_{n-1}} u_{n-\nu} \]
\[= \frac{1}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} \left( \frac{P_\nu}{\nu + 1} - \frac{P_\nu}{n + 1} \right) u_{n-\nu}. \]
For the Fourier series of \( f(t) \) at \( t = k \),
\[A_n = \frac{2}{\pi} \int_0^\pi \phi(t) \cos ntdt \]
so that
\[l_n - l_{n-1} = \frac{2}{\pi} \int_0^\pi \phi(t) \left( \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} \left( \frac{P_n}{k+1} - \frac{P_k}{n+1} \right) \frac{\cos(n-k)t}{\log(n-k+1)} \right) dt. \]
Thus in order to prove the theorem, we have to establish that
\[\sum_n \left| \int_0^\pi \phi(t) g(n, t) dt \right| < \infty, \]
where
\[g(n, t) = \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} \left( \frac{P_n}{k+1} - \frac{P_k}{n+1} \right) \frac{\cos(n-k)t}{\log(n-k+1)}. \]
We observe that
\[ \int_0^x \phi(t) g(n, t) \, dt = - \int_0^x \left( \int_0^t g(n, u) \, du \right) \, d\phi(t), \]
and
\[ \sum_n \left| \int_0^x \left( \int_0^t g(n, u) \, du \right) \, d\phi(t) \right| \leq \int_0^x \left| d\phi(t) \right| \left\{ \sum_n \left| \int_0^t g(n, u) \, du \right| \right\}. \]

Since, by hypothesis, \( \int_0^x |d\phi(t)| < \infty \), it suffices for our purpose to show that, uniformly for \( 0 < t < \pi \),
\[ \sum_n \left| \int_0^t g(n, u) \, du \right| < \infty, \]
or, what is the same thing,
\[ \sum = \sum_n \left| \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} \left( \frac{P_n}{k+1} - \frac{P_k}{n+1} \right) \frac{\sin(n-k)t}{(n-k) \log(n-k+1)} \right| < \infty. \]

Denoting \( \tau = \lceil 1/t \rceil \), we have
\[ \sum \leq \sum_1^r \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{[n/2]-1} \left( \frac{P_n}{k+1} - \frac{P_k}{n+1} \right) \frac{\sin(n-k)t}{(n-k) \log(n-k+1)} \right| + \sum_{\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{[n/2]} \left( \frac{P_n}{k+1} - \frac{P_k}{n+1} \right) \frac{\sin(n-k)t}{(n-k) \log(n-k+1)} \right|. \]

Now since \( |\sin(n-k)t| \leq (n-k)t \) and \( P_n(n+1) \geq P_k(k+1) \) for \( k \leq n \), we have
\[ \sum = \sum_1^r \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{n-1} \left( \frac{P_n}{k+1} - \frac{P_k}{n+1} \right) \frac{\sin(n-k)t}{(n-k) \log(n-k+1)} \right| \leq At \sum_1^r \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} \frac{P_n}{k+1} \]

(3.1)
\[ = At \sum_1^r \frac{P_n}{P_{n-1}} \frac{P_n}{P_{n-1}} \]
\[ = O(1). \]

By Abel’s transformation and taking \( r \) to be a fixed number \( > \pi \).
\[
\sum_{2}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{k=0}^{[n/2]-1} \left( \frac{P_n}{k+1} - \frac{P_k}{n+1} \right) \frac{\sin(n - k) t}{(n - k) \log(n - k + 1)}
\]

\[
= \sum_{r+1}^{\infty} \frac{1/(n + 1)}{P_n P_{n-1}} \left| \sum_{k=0}^{[n/2]-1} \frac{P_n(n + 1) - P_k(k + 1)}{(n - k) \log(n - k + 1)} \frac{\sin(n - k) t}{k + 1} \right|
\]

\[
= O \left[ \sum_{r+1}^{\infty} \frac{1/(n + 1)}{P_n P_{n-1}} \left( \log \frac{r}{t} \right)^{[n/2]-1} \frac{P_n(n + 1) - P_k(k + 1)}{(n - k) \log(n - k + 1)} \right]
\]

(3.2)

\[
= O \left( \log \frac{r}{t} \right) \sum_{r+1}^{\infty} \frac{1/(n + 1)}{P_n P_{n-1}} = O \left( \frac{\log r/t}{P_r} \right)
\]

\[= O(1), \]

by using Lemmas 3 and 4 (i).

Since \(1/(n + 1)(n - k) = 1/(k + 1)(n - k) - 1/(n + 1)(k + 1)\), we have

\[
\sum_{3}^{\infty} = \sum_{r+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{[n/2]-1} \left( \frac{P_n}{k+1} - \frac{P_k}{n+1} \right) \frac{\sin(n - k) t}{(n - k) \log(n - k + 1)} \right|
\]

\[
\leq \sum_{r+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{[n/2]} \frac{P_n - P_k}{(k + 1)} \frac{\sin(n - k) t}{(n - k) \log(n - k + 1)} \right|
\]

\[
+ \sum_{r+1}^{\infty} \frac{1/(n + 1)}{P_n P_{n-1}} \left| \sum_{k=0}^{[n/2]} \frac{P_k}{k + 1} \frac{\sin(n - k) t}{(n - k) \log(n - k + 1)} \right|
\]

\[= \sum_{31} + \sum_{32}, \text{ say.} \]

Now since for \(k \geq [n/2]\), \(P_n - P_k = O(1)\), we obtain

\[
\sum_{31} = \sum_{r+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{[n/2]} \frac{P_n - P_k}{k + 1} \frac{\sin(n - k) t}{(n - k) \log(n - k + 1)} \right|
\]

\[
= O \left[ \sum_{r+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{[n/2]} \frac{P_n - P_k}{(k + 1)} \frac{1}{(n - k) \log(n - k + 1)} \right| \right]
\]

(3.3)

\[
= O \left( \frac{\log \log n}{n \log^2 n} \right) = O(1).
\]
Finally since \( \sum_{a}^b \sin nt = O(1/t) \) for all \( a \) and \( b \), we have by Abel's transformation

\[
\sum_{\mathfrak{S}_2} = \sum_{r+1}^{\infty} \frac{1/(n + 1)}{P_n P_{n-1}} \left| \sum_{k=[n/2]}^{n-1} \frac{P_k}{k+1} \frac{\sin((n - k)t)}{\log(n - k + 1)} \right|
\]

\[
= O\left( \sum_{r+1}^{\infty} \frac{1/(n + 1)}{P_n P_{n-1}} \sum_{k=[n/2]}^{n-2} \frac{1}{t} \left| \Delta \left( \frac{P_k}{k+1} \frac{1}{\log(n - k + 1)} \right) \right| \right)
\]

\[
+ O\left( \frac{1}{t} \right) \left( \sum_{r+1}^{\infty} \frac{1}{P_n P_{n-1}} \frac{1}{n} \right)
\]

(3.4)

\[
= O\left( \frac{1}{t} \right) \left( \sum_{r+1}^{\infty} \frac{1}{n^2 \log n} \right)
\]

\[
= O\left( \frac{1}{t} \right) \left( \frac{1}{r \log r} \right)
\]

\[
= O(1),
\]

using Lemma 4 (ii).

Collecting (3.1), (3.2), (3.3), and (3.4), we see that the theorem is completely proved.

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References


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