ON INEQUALITIES WITH ALTERNATING SIGNS

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1. **Introduction.** In a recent paper, Olkin [2], established the following result.

THEOREM 1. *Let*

$$\text{(1)}\qquad \begin{aligned} &\text{(a)}\ a_1 \geqq a_2 \geqq \cdots \geqq a_m \geqq 0,\\ &\text{(b)}\ 1 \geqq w_1 \geqq w_2 \geqq \cdots w_m \geqq 0,\\ &\text{(c)}\ h(x) \textit{ be convex in } [0, a_1],\ h(0) \leqq 0. \end{aligned}$$

Then

$$\text{(2)}\qquad \sum_{j=1}^{m} (-1)^{j-1} w_j h(a_j) \geqq h\left(\sum_{j=1}^{m} (-1)^{j-1} w_j a_j \right).$$

This is an extension of the result given in [1], which in turn is an extension of the original result of Weinberger, [4].[1] The purpose of this paper is to show that Olkin's result is a special case of an interesting inequality due to Steffensen, [3].

2. **Steffensen's inequality.** The result of Steffensen is the following.

THEOREM 2. *Let*

$$\text{(1)}\qquad \begin{aligned} &\text{(a)}\ f(t) \textit{ be non-negative and monotone decreasing in } [a, b].\\ &\text{(b)}\ g(t) \textit{ satisfy the constraint } 0 \leqq g(t) \leqq 1,\ t \textit{ in } [a, b]. \end{aligned}$$

Then

$$\text{(2)}\qquad \int_{b-c}^{b} f(t)dt \leqq \int_{a}^{b} f(t)g(t)dt \leqq \int_{a}^{a+c} f(t)dt,$$

where

$$\text{(3)}\qquad c = \int_{a}^{b} g(t)dt.$$

Let us give a proof for the sake of completeness. Define the function $u(s)$ by the relation

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[1] This is an appropriate place to note that the results in [1] and [4] are themselves special cases of Theorem 108 of *Inequalities* by Hardy, Littlewood and Pólya.

$$\int_a^s f(t)g(t)dt = \int_a^u f(t)dt. \tag{4}$$

It is easy to see that $u(a)=a$, that $u(s)$ is continuous and monotone increasing as s goes from a to b, and that $u(s)\leqq s$. The condition that $0\leqq g(t)\leqq 1$ is essential here. We have, upon differentiating,

$$f(u)\frac{du}{ds} = f(s)g(s), \tag{5}$$

whence,

$$\frac{du}{ds} = \frac{f(s)g(s)}{f(u)} \leqq g(s), \tag{6}$$

taking account of the fact that $u(s)\leqq s$ and that $f(s)$ is monotone decreasing. Hence,

$$u \leqq a + \int_a^s g(s)ds. \tag{7}$$

This yields the right-hand side of (2), and the left-hand side is derived in the same fashion.

3. **Olkin's inequality.** To derive Olkin's result, choose for $g(t)$ the function defined by

$$g(t) = \lambda_k, \qquad a_{k+1} \leqq t \leqq a_k, \qquad k = 1, 2, \cdots, m-1, \tag{1}$$

where $\lambda_1=w_1$, $\lambda_2=w_1-w_2$, $\lambda_3=w_1-w_2+w_3$, and so on, and for $h(t)$ the function defined by

$$h'(t) = f(t). \tag{2}$$

Using the inequality of (2.2) for the preceding choice of functions, we obtain a slightly stronger result of the form of (1.2).

4. **A generalization of Steffensen's inequality.** Let us now establish a generalization of the inequality given in §2. It will be clear that many further results of this type can be obtained using the same techniques.

THEOREM 3. *Let*

(a) *$f(t)$ be non-negative and monotone decreasing in $[a, b]$.*

(1) (b) *$f\in L^p[a, b]$.*

(c) *$g(t)\geqq 0$ in $[a, b]$ and $\int_a^b g^{p'}dt\leqq 1$,* *where* $p>1$ *and* $1/p+1/p'=1$. *Then*

$$\left(\int_a^b fgdt\right)^p \leqq \int_a^{a+c} f^p dt, \tag{2}$$

where

$$c = a + \left(\int_a^b gdt\right)^p. \tag{3}$$

PROOF. Consider the function $u(t)$ defined for $a \leqq t \leqq b$ by the equation

$$\left(\int_a^t fgdt\right)^p = \int_a^u f^p dt. \tag{4}$$

Since

$$\left(\int_a^t fgdt\right)^p \leqq \left(\int_a^t f^p dt\right)\left(\int_a^t g^{p'} dt\right)^{p/p'} \leqq \int_a^t f^p dt, \tag{5}$$

we see that $u(t)$ exists and satisfies the relation $u(t) \leqq t$ for t in $[a, b]$ with $u(a)=a$. This function is monotone increasing and satisfies the differential equation

$$f(u)^p \frac{du}{dt} = pf(t)g(t)\left(\int_a^t fgdt\right)^{p-1} \tag{6}$$

almost everywhere.

The monotonic nature of $f(t)$ and $u(t)$ yield the inequality

$$\frac{du}{dt} \leqq pg(t)\left(\int_a^t gdt\right)^{p-1}, \tag{7}$$

whence

$$u(t) \leqq a + \left(\int_a^t gdt\right)^p. \tag{8}$$

This completes the proof.

REFERENCES

1. R. Bellman, *On an inequality of Weinberger*, Amer. Math. Monthly vol. 60 (1953) p. 402.

2. I. Olkin, *An inequality of Weinberger and Bellman*, Proc. Nat. Acad. Sci. U.S.A. vol. 45 (1959) pp. 230–231.

3. J. F. Steffensen, *Bounds of certain trigonometric integrals*, Tenth Scandinavian Math. Congress, 1946, pp. 181–186.

4. H. F. Weinberger, *An inequality with alternating signs*, Proc. Nat. Acad. Sci. U.S.A. vol. 38 (1952) pp. 611–613.

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