ON INEQUALITIES WITH ALTERNATING SIGNS

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1. Introduction. In a recent paper, Olkin [2], established the following result.

**Theorem 1.** Let

(a) \( a_1 \geq a_2 \geq \cdots \geq a_m \geq 0, \) 

(b) \( 1 \geq w_1 \geq w_2 \geq \cdots w_m \geq 0, \) 

(c) \( h(x) \) be convex in \([0, a_1], h(0) \leq 0.\)

Then

\[ \sum_{j=1}^{m} (-1)^{i-1} w_j h(a_j) \geq h \left( \sum_{j=1}^{m} (-1)^{i-1} w_j a_j \right). \]

This is an extension of the result given in [1], which in turn is an extension of the original result of Weinberger, [4]. The purpose of this paper is to show that Olkin’s result is a special case of an interesting inequality due to Steffensen, [3].

2. Steffensen’s inequality. The result of Steffensen is the following.

**Theorem 2.** Let

(a) \( f(t) \) be non-negative and monotone decreasing in \([a, b].\)

(b) \( g(t) \) satisfy the constraint \( 0 \leq g(t) \leq 1, t \in [a, b]. \)

Then

\[ \int_{b-c}^{b} f(t) \, dt \leq \int_{a}^{b} f(t) g(t) \, dt \leq \int_{a}^{a+c} f(t) \, dt, \]

where

\[ c = \int_{a}^{b} g(t) \, dt. \]

Let us give a proof for the sake of completeness. Define the function \( u(s) \) by the relation

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This is an appropriate place to note that the results in [1] and [4] are themselves special cases of Theorem 108 of *Inequalities* by Hardy, Littlewood and Pólya.
\[ \int_{a}^{b} f(t)g(t)\,dt = \int_{a}^{u} f(t)\,dt. \]

It is easy to see that \( u(a) = a \), that \( u(s) \) is continuous and monotone increasing as \( s \) goes from \( a \) to \( b \), and that \( u(s) \leq s \). The condition that \( 0 \leq g(t) \leq 1 \) is essential here. We have, upon differentiating,

\[ f(u) \frac{du}{ds} = f(s)g(s), \]

whence,

\[ \frac{du}{ds} = \frac{f(s)g(s)}{f(u)} \leq g(s), \]

taking account of the fact that \( u(s) \leq s \) and that \( f(s) \) is monotone decreasing. Hence,

\[ u \leq a + \int_{a}^{s} g(s)\,ds. \]

This yields the right-hand side of (2), and the left-hand side is derived in the same fashion.

3. Olkin's inequality. To derive Olkin's result, choose for \( g(t) \) the function defined by

\[ g(t) = \lambda_k, \quad a_{k+1} \leq t \leq a_k, \quad k = 1, 2, \ldots, m - 1, \]

where \( \lambda_1 = w_1, \lambda_2 = w_1 - w_2, \lambda_3 = w_1 - w_2 + w_3, \) and so on, and for \( h(t) \) the function defined by

\[ h'(t) = f(t). \]

Using the inequality of (2.2) for the preceding choice of functions, we obtain a slightly stronger result of the form of (1.2).

4. A generalization of Steffensen's inequality. Let us now establish a generalization of the inequality given in §2. It will be clear that many further results of this type can be obtained using the same techniques.

**Theorem 3.** Let

(a) \( f(t) \) be non-negative and monotone decreasing in \( [a, b] \).

(b) \( f \in L^p[a, b] \).

(c) \( g(t) \geq 0 \) in \( [a, b] \) and \( \int_{a}^{b} g^p\,dt \leq 1 \),
where $p > 1$ and $1/p + 1/p' = 1$. Then

$$\left( \int_a^b f g dt \right)^p \leq \int_a^{a+c} f^p dt,$$

where

$$c = a + \left( \int_a^b g dt \right)^p.$$

Proof. Consider the function $u(t)$ defined for $a \leq t \leq b$ by the equation

$$\left( \int_a^t f g dt \right)^p = \int_a^t f^p dt.$$

Since

$$\left( \int_a^t f g dt \right)^p \leq \left( \int_a^t f^p dt \right) \left( \int_a^t g^p' dt \right)^{p/p'},$$

we see that $u(t)$ exists and satisfies the relation $u(t) \leq t$ for $t$ in $[a, b]$ with $u(a) = a$. This function is monotone increasing and satisfies the differential equation

$$f(u)^p \frac{du}{dt} = pf(t)g(t) \left( \int_a^t f g dt \right)^{p-1}$$

almost everywhere.

The monotonic nature of $f(t)$ and $u(t)$ yield the inequality

$$\frac{du}{dt} \leq pg(t) \left( \int_a^t g dt \right)^{p-1},$$

whence

$$u(t) \leq a + \left( \int_a^t g dt \right)^p.$$

This completes the proof.

References