THE BERNSTEIN PROBLEM

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Let $w(x)$ be a positive valued, continuous function of real x, such that for each $n=0, 1, 2, \cdots$, $x^n w(x)$ is bounded. S. Bernstein asks for conditions on $w(x)$ that the weighted polynomials $P(x)w(x)$ be uniformly dense in the continuous complex valued functions which vanish at infinity (Pollard [5]).

THEOREM. *A necessary and sufficient condition that the weighted polynomials $P(x)w(x)$ fail to be dense in the continuous functions which vanish at infinity is that there be an entire function $F(z)$ of exponential type, not a polynomial, which is real for real z and whose zeros λ_n are real and simple, such that*

$$\int \frac{\log^+ |F(t)|}{1+t^2}\, dt < \infty$$

and

$$\sum |F'(\lambda_n)w(\lambda_n)|^{-1} < \infty.$$

LEMMA 1. *If μ is a nonzero measure on the Borel sets of the real line, such that for each $n=0, 1, 2, \cdots$*

$$\int |t^n d\mu(t)| < \infty$$

and

$$\int t^n d\mu(t) = 0,$$

and if

$$M(z) = \sup |P(z)|$$

where $P(z)$ ranges in the polynomials such that

$$\int |P(t)d\mu(t)| \leqq 1,$$

then $M(z)$ is bounded away from zero, and

$$\int \frac{\log M(t)}{1+t^2}\, dt < \infty,$$

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and

$$\log M(x+iy) \leqq \frac{|y|}{\pi}\int \frac{\log M(t)}{(t-x)^2+y^2}\,dt \qquad (y \neq 0).$$

If $G(z)$ is an entire function of minimal exponential type such that $\int |G(t)d\mu(t)| < \infty$, and

$$G(iy) = o(M(iy)) \qquad (|y| \to \infty), \tag{1}$$

then

$$\int G(t)d\mu(t) = 0.$$

LEMMA 2. *Let $F(z)$ be a nonconstant entire function of exponential type, whose zeros λ_n are real and simple, such that*

$$\int \frac{\log^+ |F(t)|}{1+t^2}\,dt < \infty. \tag{2}$$

Let $G(z)$ be an entire function of exponential type such that

$$\int \frac{\log^+ |G(t)|}{1+t^2}\,dt < \infty$$

$$\sum \left| \frac{G(\lambda_n)}{F'(\lambda_n)\lambda_n} \right| < \infty,$$

and

$$G(iy) = o(F(iy)) \qquad (|y| \to \infty),$$

then

$$\frac{G(z)}{F(z)} = \frac{G(\lambda_n)}{F'(\lambda_n)(z-\lambda_n)}.$$

LEMMA 3. *If $F(z)$ and $G(z)$ satisfy the hypotheses of Lemma 2 and if*

$$\sum \left| \frac{G(\lambda_n)}{F'(\lambda_n)} \right| < \infty$$

and

$$yG(iy) = o(F(iy)) \qquad (|y| \to \infty),$$

then

$$\sum \frac{G(\lambda_n)}{F'(\lambda_n)} = 0.$$

PROOF OF LEMMA 1. Let μ be as in the statement of the lemma. By the lemma of Pollard [5, p. 407], for every polynomial $P(z)$

$$P(z) \int \frac{d\mu(t)}{t-z} = \int \frac{P(t)d\mu(t)}{t-z}.$$

The proof of the properties of $M(z)$ stated in the lemma is in [2, pp. 149–150]. Let $G(z)$ be as in the statement of the lemma and let

$$H(z) = \int \frac{G(t)-G(z)}{t-z}\, d\mu(t).$$

As in [2, pp. 147–148], $H(z)$ is an entire function of minimal exponential type. For every polynomial $P(z)$ such that $\int |P(t)d\mu(t)| \leqq 1$,

$$\begin{aligned} |H(z)| &\leqq \left|\int (t-z)^{-1}G(t)d\mu(t)\right| + |G(z)| \left|\int (t-z)^{-1}d\mu(t)\right| \\ &\leqq \left|\int (t-z)^{-1}G(t)d\mu(t)\right| + |G(z)/P(z)| \left|\int (t-z)^{-1}P(t)d\mu(t)\right| \\ &\leqq |y|^{-1}\left(\int |G(t)d\mu(t)| + |G(z)/P(z)|\right). \end{aligned}$$

By the definition of $M(z)$,

$$|H(z)| \leqq |y|^{-1}\left(\int |G(t)d\mu(t)| + |G(z)|/M(z)\right).$$

By (1), $H(z)$ goes to zero at both ends of the imaginary axis. By Boas [1, p. 83], $H(z) \equiv 0$. Therefore,

$$\begin{aligned} \int G(t)d\mu(t) &= \int \frac{tG(t)-zG(t)}{t-z}\, d\mu(t) \\ &= \int \frac{tG(t)-zG(z)}{t-z}\, d\mu(t). \end{aligned}$$

For every polynomial $P(z)$ such that $\int |P(t)d\mu(t)| \leqq 1$,

$$\begin{aligned} \left|\int G(t)d\mu(t)\right| &\leqq \left|\int \frac{tG(t)d\mu(t)}{t-z}\right| + \left|\frac{zG(z)}{P(z)} \int \frac{P(t)d\mu(t)}{t-z}\right| \\ &\leqq \left|\int \frac{tG(t)d\mu(t)}{t-z}\right| + |y|^{-1}|zG(z)/P(z)|. \end{aligned}$$

By the definition of $M(z)$,

$$\left|\int G(t)d\mu(t)\right| \leqq \left|\int \frac{tG(t)d\mu(t)}{t-z}\right| + |y|^{-1}|zG(z)|/M(z).$$

Let $z=iy$ where $y\to+\infty$. The first term goes to zero by the Lebesgue dominated convergence theorem. The second term goes to zero by the hypothesis (1) on $G(z)$. Therefore, $\int G(t)d\mu(t)=0$. Q.E.D.

PROOF OF LEMMA 2. Let $F(z)$ and $G(z)$ satisfy the hypotheses of the lemma. Let

$$G_1(z) = F(z)\sum \frac{G(\lambda_n)}{F'(\lambda_n)(z-\lambda_n)},$$

$$G_2(z) = G(z) - G_1(z).$$

It is obvious that $G_1(z)$ is an entire function and that

$$G_1(z) \leqq 2|y|^{-1}|zF(z)|\sum\left|\frac{G(\lambda_n)}{F'(\lambda_n)\lambda_n}\right|$$

where the hypothesis is that the sum converges. As in [2, pp. 147–148], $G_1(z)$ has exponential type and the hypothesis (2) implies that

$$\int \frac{\log^+|G_1(t)|}{1+t^2}\,dt < \infty.$$

By the Lebesgue dominated theorem, $G(iy)=o(F(iy))$ $(|y|\to\infty)$. Because of the hypotheses on $G(z)$, $G_2(z)$ is an entire function of exponential type such that

$$\int \frac{\log^+|G_2(t)|}{1+t^2}\,dt < \infty \tag{3}$$

and

$$G_2(iy) = o(F(iy)) \qquad (|y|\to\infty). \tag{4}$$

Since $G_2(\lambda_n)=0$ for all n and the zeros of $F(z)$ are all simple, $H(z)=G_2(z)/F(z)$ is an entire function. By (2) and (3) and the representation theorem for functions analytic in a half plane (Boas [1, p. 92)], $H(z)$ has exponential type and

$$\int \frac{\log^+|H(t)|}{1+t^2}\,dt < \infty.$$

By Boas [2, p. 97], the indicator diagram of $H(z)$ is a vertical line

segment. By (4), $H(z)$ is an entire function of minimal exponential type which goes to zero at both ends of the imaginary axis. By Boas [1, p. 83], $H(z) \equiv 0$. Q.E.D.

PROOF OF LEMMA 3. Since $F(z)$ and $G(z)$ satisfy the hypotheses of Lemma 2,

$$\int \frac{G(t) - G(z)}{t - z} d\mu(t) = 0.$$

Since $F(z)$ and $zG(z)$ satisfy the hypotheses of Lemma 2,

$$\int \frac{tG(t) - zG(z)}{t - z} d\mu(t) = 0.$$

So

$$\int G(t) d\mu(t) = \int \frac{tG(t) - zG(z)}{t - z} d\mu(t) - z \int \frac{G(t) - G(z)}{t - z} d\mu(t)$$
$$= 0. \qquad \text{Q.E.D.}$$

PROOF OF THEOREM, THE SUFFICIENCY. Let $F(z)$ be as in the statement of the theorem. If $P(z)$ is any polynomial,

$$\int \frac{\log^+ |P(t)|}{1 + t^2} dt < \infty$$

and elementary estimates from the Hadamard factorization of $F(z)$ show that

$$yP(iy) = o(F(iy)) \qquad (|y| \to \infty).$$

Let μ be the Borel measure with its mass concentrated at the zeros of $F(z)$ and with mass $(F'(\lambda_n)w(\lambda_n))^{-1}$ at λ_n. The hypothesis is that $\int |d\mu(t)| < \infty$. By Lemma 3, for every polynomial $P(z)$,

$$\int P(t)w(t)d\mu(t) = \sum \frac{P(\lambda_n)}{F'(\lambda_n)} = 0.$$

Since μ is not the zero measure, the weighted polynomials are not uniformly dense. Q.E.D.

PROOF OF THEOREM, THE NECESSITY. Suppose the weighted polynomials are not dense. Apply the lemma of [4] with S the real line and E the set of weighted polynomials. In the notation of [4], $U(E)$ contains a nonzero extreme point μ and if $g(t)$ is a Borel measurable function of real t such that

$$\int \left| g(t)w(t)d\mu(t) \right| < \infty$$

and

$$\int g(t)w(t)d\mu(t) = 0,$$

then there is a sequence $P_n(z)$ of polynomials such that $\lim_n \int \left| g(t) - P_n(t) \right| w(t) \left| d\mu(t) \right| = 0$. Let $M(z)$ be defined for μ as in Lemma 1. By this definition, the sequence $P_n(z)/M(z)$ converges uniformly in the complex plane. By properties of $M(z)$ concluded in Lemma 1, the limit is a function of the form $G(z)/M(z)$ where $G(z)$ is an entire function of minimal exponential type. It is clear that $g(t) = G(t)$ a.e. with respect to μ.

Since $w(x) > 0$ for all x, it is obvious that the support of μ contains more than one point. Let $[a, b]$ be any finite interval which contains at least two points of the support of μ. Then certainly we can choose a Borel measurable function $g(t)$ which vanishes a.e. outside of $[a, b]$, such that

$$\int \left| g(t) \right| w(t) \left| d\mu(t) \right| = 1$$

and

$$\int g(t)w(t)d\mu(t) = 0.$$

As we have seen, $g(t)$ is equal a.e. with respect to μ to an entire function $G(z)$. Therefore, the support of μ is contained entirely in the set of zeros of $G(z)$ and the interval $[a, b]$. By the arbitrariness of $[a, b]$, the support of μ is a discrete set $\{\lambda_n\}$.

Let λ_1 and λ_2 be any two distinct points of the support of μ and let $g(t)$ be the function which vanishes everywhere except at λ_1 and at λ_2, such that

$$g(\lambda_1) = [(\lambda_1 - \lambda_2)w(\lambda_1)\mu(\{\lambda_1\})]^{-1},$$
$$g(\lambda_2) = [(\lambda_2 - \lambda_1)w(\lambda_2)\mu(\{\lambda_2\})]^{-1}.$$

Then

$$\int \left| g(t) \right| w(t) \left| d\mu(t) \right| = 2 \left| \lambda_2 - \lambda_1 \right|^{-1} < \infty$$

and

$$\int g(t)w(t)d\mu(t) = 0.$$

Let $G(z)$ be the entire function constructed above corresponding to $g(t)$. Since $G(z)/M(z)$ is a uniform limit of continuous functions which vanish at infinity in the complex plane,

$$G(iy) = o(M(iy)) \qquad (|y| \to \infty).$$

Let $F(z) = (z-\lambda_1)(z-\lambda_2)G(z)$. Since $G(z)$ is dominated in the complex plane by $M(z)$ where $M(z)$ is bounded away from zero and

$$\int \frac{\log M(t)}{1+t^2}\, dt < \infty,$$

it follows that

$$\int \frac{\log^+ |G(t)|}{1+t^2}\, dt < \infty$$

and

$$\int \frac{\log^+ |F(t)|}{1+t^2}\, dt < \infty.$$

Let λ_3 be any third point of the support of μ and let $H(z) = (z-\lambda_1)^{-1}(z-\lambda_3)^{-1}F(z)$. Then $H(z)$ is an entire function of minimal exponential type, $\int |H(t)|w(t)|d\mu(t)| < \infty$ and

$$H(iy) = o(M(iy)) \qquad (|y| \to \infty).$$

By Lemma 1,

$$\begin{aligned}\int H(t)w(t)d\mu(t) &= H(\lambda_1)w(\lambda_1)\mu(\{\lambda_1\}) + H(\lambda_3)w(\lambda_3)\mu(\{\lambda_3\}) \\ &= 0.\end{aligned}$$

Since

$$\begin{aligned}H(\lambda_1) &= [(\lambda_1 - \lambda_3)w(\lambda_1)\mu(\{\lambda_1\})]^{-1}, \\ H(\lambda_3) &= [(\lambda_3 - \lambda_1)w(\lambda_3)\mu(\{\lambda_3\})]^{-1}.\end{aligned}$$

To summarize, if λ is any real zero of $F(z)$ in the support of μ, $F'(\lambda)w(\lambda)\mu(\{\lambda\}) = 1$.

We claim that $F(z)$ has no other zeros. For suppose $F(z)$ had a zero λ which was not in the support of μ. Let λ_1 be a zero of $F(z)$ in the support of μ. Let $L(z) = (z-\lambda)^{-1}(z-\lambda_1)^{-1}F(z)$. It follows by the use of Lemma 3 as above that $\int L(t)w(t)d\mu(t) = 0$, or equivalently that

$$(\lambda_1 - \lambda)^{-1} F'(\lambda_1) w(\lambda_1) \mu(\{\lambda_1\}) = 0,$$

which contradicts the fact that

$$F'(\lambda_1) w(\lambda_1) \mu(\{\lambda_1\}) = 1.$$

In other words, the zeros of $F(z)$ are real and simple, and are in one-to-one correspondence with the points of support of μ. It is obvious from the properties of μ that μ is supported at more than a finite number of points and that therefore $F(z)$ is not a polynomial. But now

$$\sum \left| F'(\lambda_n) w(\lambda_n) \right|^{-1} = \int \left| d\mu \right| = 1. \qquad \text{Q.E.D.}$$

References

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