# ALGEBRAIC VECTOR BUNDLES OVER THE PRODUCT OF AN AFFINE CURVE AND THE AFFINE LINE

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Let $k$ be an *algebraically closed field* and $A$ the affine algebra of a nonsingular irreducible affine curve defined over $k$. Then $A[X]$, the algebra of polynomials over $A$, is the affine algebra of the product variety $C\times k^1$. Let $P$ be a projective module of finite type over $A[X]$. The module $P$ corresponds to an algebraic vector bundle over $C\times k^1$ [**4**, §4, Chapter II]. The purpose of this note is to prove the following

THEOREM. *There exists a projective module $\Lambda$ over $A$ of finite type, such that*

$$P = \Lambda \otimes_A A[X]$$

(*i.e. every algebraic vector bundle over $C\times k^1$ can be obtained as the inverse image of an algebraic vector bundle over $C$, by the projection mapping $C\times k^1\to k^1$*).

This result generalizes the one proved in [**6**] that every algebraic vector bundle over $k^2$ is trivial. The method of proof is also similar to [**6**].

Let $B$ be a normal (integrally closed) noetherian ring. Then a divisor $D$ in $B$ [**3**, §2, Chapter III] is a finite linear combination with integral coefficients of prime ideals of height 1 (minimal prime ideals). Let $S$ be a multiplicatively closed subset in $B$, not containing 0 and containing 1, and $BS^{-1}$ the ring of classes of fractions $b/s$, $b\in B$, $s\in S$. Then for a divisor $D=\sum_{i\in I} a_i p_i$ in $B$ ($p_i$ prime, $a_i$ integer) we define a divisor $DS^{-1}$ in $BS^{-1}$ by

$$DS^{-1} = \sum_{j\in J} a_j(p_jS^{-1}),$$

$J$ being the subset of $I$ containing those indices for which $p_i$ does not intersect $S$.

PROPOSITION 1. *Every divisor $D'$ in $BS^{-1}$ can be put in the form $DS^{-1}$ where $D$ is a divisor in $B$.*

We have only to prove that every prime ideal $p'$ of height 1 in $BS^{-1}$ can be expressed in the form $pS^{-1}$ where $p$ is a prime ideal of

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height 1 in $B$. This is an immediate consequence of [**2**, Proposition 1, Exposé 1].

PROPOSITION 2. *Let $B$ further satisfy the condition, that for every maximal ideal $m$, the localized ring $B_m$ (i.e. $B_m = BS^{-1}$ with $S = B - m$) is factorial. Then for every projective module $N$ over $BS^{-1}$ of rank* 1, *there exists a projective module $M$ over $B$ of rank* 1, *such that $N$ is isomorphic to $M \otimes_B BS^{-1}$.*

$N$ is isomorphic to an inversible ideal (cf. [**1**, §3, Chapter VII]) in $BS^{-1}$. Let $\mathfrak{A}$ be such an ideal. It is easy to verify that $(BS^{-1}:(BS^{-1}:\mathfrak{A})) = \mathfrak{A}$. Therefore $\mathfrak{A}$ is the ideal of a divisor $D'$ in $BS^{-1}$ (cf. [**3**, Corollary, Theorem 4, §2, Chapter III]). By Proposition 1, $D'$ can be "lifted" to a divisor $D$ in $B$. The ideal $I(D)$ of the divisor $D$ is "locally" principal because of the hypothesis that $B_m$ is factorial for every maximal ideal $m$ (cf. [**3**, Theorem 1, §3, Chapter III]). Therefore $I(D)$ is an inversible ideal and corresponds to a projective module $M$ over $B$ of rank 1 (cf. [**5**, Proposition 3]). It is easily verified that $N$ is isomorphic to $M \otimes_B BS^{-1}$.

COROLLARY. *Proposition 2 is valid when $B$ is the affine algebra of a nonsingular irreducible affine variety.*

For, in this case, it is well-known that every $B_m$ is factorial.

Let now $A$ be the affine algebra of an irreducible nonsingular affine curve $C$ defined over an *algebraically closed field* $k$ and $P$ a projective module of finite type over $B = A[X]$. Then $P$ corresponds to an algebraic vector bundle over $C \times k^1$. In order to prove the theorem it can be assumed that $P$ is of rank $\leqq 2$, since by a theorem of Serre [**5**, Theorem 1], $P = L_1 + P_1$, where $L_1$ is free and $P_1$ is of rank $\leqq 2$. Let $S$ be $k[X]^*$, the set of all nonzero polynomials over $k$. When $P$ is of rank 1, the theorem stated is well-known. Therefore, we suppose hereafter that $P$ is of rank 2. Consider $PS^{-1}$. This is a projective module over $BS^{-1}$ (the affine algebra of the curve $C$ extended to $K = k(X)$) of rank 2. Then $BS^{-1}$ is a Dedekind ring, so that $P \otimes_{k[X]} k(X) = PS^{-1} = N_1 + N_2$, where $N_1$ is a free module over $BS^{-1}$ of rank 1 and $N_2$ a projective module over $BS^{-1}$ of rank 1 (cf. [**5**, Proposition 7]). We can choose elements $y_1, z_1, \cdots, z_p$ of $P$ and $a \in k[X]^*$ such that $y_1 \otimes 1/a$ generates $N_1$ and $z_1 \otimes 1/a, \cdots, z_p \otimes 1/a$ generate $N_2$. Because of Proposition 2 it can be supposed that $z_1, \cdots, z_p$ generate a projective submodule $M_2$ of $P$ of rank 1. Let $M_1$ be the free submodule of $P$ of rank 1, generated by $y_1$. Then $L = M_1 + M_2$ is a submodule of $P$ satisfying the following conditions:

(i) $L$ is a direct sum of projective modules of rank 1;

(ii) $aP \subset L \subset P$, $a \in k[X]^*$.

Further it can be supposed that

(iii) $(L:P)=(a)$, and

(iv) If $L_1$ is a submodule of $P$ satisfying (i) and (iii) in place of $L$ and $L \subset L_1 \subset P$, then it should follow that $L=L_1$.

If "$a$" were a unit, the required theorem is proved, otherwise $\exists$ a prime $p$ ($p \neq 1$) of $k[X]$ which divides $a$. Let $a=a_1p$. Because of the property $L: P=(a)$ on account of (iii), $\exists$ an element $x \in P$ such that $a_1x \notin L$ and $pa_1x \in L$. This shows that $L \cap pP \neq pL$. Now consider the homomorphism

$$\Pi: L/pL \to P/pP$$

induced by the inclusion $L \subset P$. Then $P$ gives rise to the following exact sequence

$$0 \to \operatorname{Ker} \Pi \to L/pL \to \operatorname{Im} \Pi \to 0 \tag{1}$$

of modules over the ring $(k[X]/(p)) \otimes_{k[X]} A[X]$ which is canonically isomorphic to $A$, since $k$ is *algebraically closed*. By means of this isomorphism, we consider the modules in the exact sequence (1) as modules over $A$. Then $\operatorname{Im} \Pi$ is a projective module over $A$, since it is a submodule of $P/pP$ which is a projective module over the Dedekind ring $A$. Therefore (1) splits and we have

$$L/pL = \operatorname{Ker} \Pi + \operatorname{Im} \Pi. \tag{2}$$

Since $L$ is a direct sum of projective modules of rank 1 and the theorem stated is known to be true for projective modules of rank 1, $L$ is isomorphic to $(L/pL) \otimes_A A[X]$. Therefore the decomposition (2) can be "lifted" to the following decomposition as modules over $A[X]$:

$$L = M + N \text{ with } M/pM = \operatorname{Ker} \Pi, \qquad N/pN = \operatorname{Im} \Pi.$$

Since $L \cap pP \neq pL$, $\operatorname{Ker} \Pi \neq 0$. Therefore, since $L/pL$ is of rank 2, either $\operatorname{Ker} \Pi$ is of rank 2 and $\operatorname{Im} \Pi$ is 0, or $\operatorname{Ker} \Pi$ and $\operatorname{Im} \Pi$ are both of rank 1 respectively. In the first case, $\exists$ a submodule $L_1$ of $P$ such that $pL_1=L$. Then $L_1$ is obviously a direct sum of projective modules of rank 1 and

$$a_1P \subset L_1 \subset P, \qquad a_1p = a.$$

In the second case $\exists$ a projective submodule $M_1$ of $P$ (of rank 1) such that $pM_1=M$. It is trivial to verify that $M_1$ contains $M$ strictly. Let $L_1=M_1+N$. Then $L_1$ is a direct sum of projective modules of rank 1 and contains $L$ strictly, so that by (iv)

$$bP \subset L_1 \subset P.$$

$b$ being proper divisor of $a$.

Thus, in any case when $a$ is not a unit, we obtain $\theta P \subset L_1 \subset P$, $\theta$ being a proper divisor of $a$ and $L_1$ being a direct sum of projective modules of rank 1. By repeating the process, we find that $P$ is itself a direct sum of modules of rank 1 and as the theorem is known to be true for projective modules of rank 1, it is proved.

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