A CLASS OF IRREDUCIBLE SYSTEMS OF GENERATORS FOR INFINITE SYMMETRIC GROUPS

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If \( G \) is a group and \( M \) a subset of \( G \) then \( \{ M \} \) is the smallest subgroup of \( G \) containing \( M \). If \( \{ M \} = G \) then \( M \) is a system of generators for \( G \). If no proper subset of \( M \) is a system of generators for \( G \) then \( M \) is irreducible.

Let \( N \) be the set of positive integers; \( d \) the cardinal number of \( N \); \( d^+ \) the successor of \( d \); \( S(d, d^+) \) the group of all one-to-one mappings of \( N \) onto itself; \( A(d, d) \) the alternating subgroup of \( S(d, d^+) \); \( S(d, d) \) the finite symmetric subgroup of \( S(d, d^+) \).

**Theorem 1.** Let \( M \) consist of the sequence of odd length cycles of \( S(d, d^+) \)

\[ (1, 2, \ldots, n_1), (n_1, n_1 + 1, \ldots, n_2), \ldots, \]

\[ (n_i, n_i + 1, \ldots, n_{i+1}), \ldots \]

with the order of the cycles \( s_i = k_i \geq 3 \). Then \( M \) is an irreducible system of generators for \( A(d, d) \).

**Proof.** It is clear from the nature of the set \( M \) that \( \{ M \} \subseteq A(d, d) \). Furthermore, if \( c_i \) is removed from \( M \) then every element of the group generated by the remaining set leaves the integer \( n_{i-1} + 1 \) fixed. It is sufficient, therefore, to prove that every element of \( A(d, d) \) belongs to \( \{ M \} \). Since the 3-cycles generate \( A(d, d) \) we shall show any 3-cycle belongs to \( M \).

Let \( x_1 < x_2 < x_3 \) be any triple of elements of \( N \). There exists an element \( s_i \) of \( M \) such that \( x_i \subseteq s_i \) and \( x_i \) is not the greatest element of \( s_i \), \( i = 1, 2, 3 \). Furthermore, there exists a positive integer \( \alpha_i \) such that \( s_i^{\alpha_i}(x_i) = m_i \) where \( m_i \) is the largest integer in \( s_i \). In the set \( M \) choose the cycle, say \( s_0 \), which is the immediate successor of \( s_3 \) in the sequence of cycles of \( M \). Denote by \( s_{i1}, s_{i2}, \ldots, s_{i\alpha_i} \) the elements of \( M \) which occur in the sequence between \( s_i \) and \( s_0 \). Consider the product

\[
\left( ^{a_i-1} s_i \right) ^{a_i-1} s_i \ s_{i1}^{-1} s_{i2}^{-1} \cdots s_{i\alpha_i}^{-1} s_0 s_{i\alpha_i} \cdots s_{i1} s_{i2} s_i^{t_i-a_i}
\]

where \( t_i \) is the order of \( s_i \). A computation shows that this product is

\[ (x_1, a_2, \ldots, a_p) \]

where \( s_0 = (a_1, a_2, \ldots, a_p) \). Denote by \( d_1, d_2, d_3 \) the three cycles that

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the above formula yields. Now compute $d_1d_2^{-1}$ and $d_1d_3^{-1}$ which yield $(x_1, x_2, a_p)$ and $(x_1, x_3, a_p)$. A final computation of $d_1d_2^{-1}d_3d_1^{-1}$ shows that $(x_1, x_2, x_3)$ belongs to $M$.

**Theorem 2.** Let $M$ consist of the sequence of cycles of $S(d, d^+)$, where $c_1$ is of even length,

$$(1, 2, \cdots, n_1), (n_1, n_1 + 1, \cdots, n_2), \cdots,
(n_i, n_i + 1, \cdots, n_{i+1}), \cdots$$

with the order of the cycles $s_i = k_i \geq 4$. Then $M$ is an irreducible system of generators for $S(d, d)$.

**Proof.** By an argument similar to the one given above, it is clear that $A(d, d) \subseteq M$. If $x_1, x_2$ are any elements of $N$ and $c_1 = (1, 2, \cdots, n_1)$ then $(x_1, x_2)c_1$ is a member of $A(d, d)$, hence in $\{M\}$. But $c_1$ belongs to $M$, hence to $\{M\}$ and $(x_1, x_2)c_1c_1^{-1} = (x_1, x_2)$ is in $\{M\}$.

**Corollary.** There exists $d^d$ irreducible systems of generators for $S(d, d)$ and $A(d, d)$.

**Theorem 3.** Let $M$ consist of all elements of the form $(i, i + 1)$, $i = 1, 2, \cdots, n, \cdots$. Then $M$ is an irreducible system of generators for $S(d, d)$.

**Proof.** Let $r < s$ be any distinct elements of $N$. Then the formula

$$(r, r + 1)(r + 1, r + 2) \cdots (s - 1, s)(s - 2, s - 1) \cdots$$

shows that $M$ contains any transposition. The set $M$ is irreducible because if $M_1$ is $M$ with $(i, i+1)$ removed then $M_1$ does not contain $(i+1, x)$ for $x > i+1$.

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