ON CUBIC FORMS PERMITTING COMPOSITION

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Let $A$ be a finite-dimensional separable alternative algebra over a field $F$. Then $A = A_1 \oplus \cdots \oplus A_r$, where each simple ideal $A_i$ is central simple over its center $Z_i$, and $Z_i$ is a separable extension of $F$ of degree $d_i$. Take $A_i$ to be associative for $i = 1, \cdots, s$ ($0 \leq s \leq r$) and not associative otherwise. The algebras $A_i$ ($i = 1, \cdots, s$) have dimension $m_i^2$ over $Z_i$, and it is well-known [1, §8.11] that the principal norm $n_i(x_i)$ for $x_i$ in $A_i$ is a (homogeneous) form of degree $m_i d_i$ over $F$ satisfying $n_i(x_i y_i) = n_i(x_i) n_i(y_i)$. The Cayley algebras $A_{s+i}$ ($i = 5 + 1, \cdots, r$) are of degree $m_i = 2$ and dimension 8 over $Z_i$, and there is similarly a norm $n_i(x_i y_i) = n_i(x_i) n_i(y_i)$. Let $x = x_1 + \cdots + x_r$ for $x_i$ in $A_i$, and define

$$N(x) = [n_1(x_1)]^{f_1} \cdots [n_r(x_r)]^{f_r}$$

for arbitrary positive integers $f_i$. Then $N(x)$ is a form of degree $n$ where

$$n = \sum_{i=1}^{r} f_i m_i d_i \quad (m_i = 2 \text{ for } i = s + 1, \cdots, r).$$

Also $N(x)$ permits composition. That is,

$$N(xy) = N(x) N(y).$$

The dimension of $A$ over $F$ is

$$\text{dim } A = \sum_{i=1}^{r} m_i^2 d_i + 8 \sum_{i=s+1}^{r} d_i.$$

Ignoring the question of inseparability by assuming characteristic $\neq 2$, we may state the principal fact about quadratic forms permitting composition [2; 7; 6] as follows. Let $A$ be a nonassociative algebra (of possibly infinite dimension) over $F$ of characteristic $\neq 2$, and assume$^2$ that $A$ has a unity element $1$. If $N(x)$ is a nondegenerate quadratic form on $A$ permitting composition, then $A$ is a finite-dimensional separable alternative algebra over $F$ and $N(x)$ is given by

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$^2$ An easy modification suffices in case $A$ has no unity element [7, p. 957; 6, p. 56]. See also the remark at the end of this paper.
(1) with \( n = 2 \) in (2). This limits the possibilities severely, and by (4) one has the classical restriction: the dimension of \( A \) is 1, 2, 4, or 8. Conversely, any such quadratic norm form is nondegenerate.

A plausible conjecture is that any (possibly infinite-dimensional) algebra \( A \) with 1 over \( F \) of characteristic 0 or \( p > n \), on which a non-degenerate\(^2\) form \( N(x) \) of degree \( n \) and permitting composition is defined, is a finite-dimensional separable alternative algebra with \( N(x) \) given by (1). Then (2) and (4) would give the restrictions on the structure and dimension of \( A \).

In this paper we study cubic forms permitting composition, and prove the following

**Theorem.** Let \( A \) be a finite-dimensional nonassociative algebra with 1 over \( F \) of characteristic \( \neq 2, 3 \). A necessary and sufficient condition for the existence of a nondegenerate cubic form \( N(x) \) on \( A \) permitting composition is that \( A \) be a separable alternative algebra for which \( N(x) \) is given by (1) with \( n = 3 \) in (2); that is, one of:

(i) \( F_1 \) (with \( f_1 = 3 \) in (1)),
(ii) a cubic field over \( F \),
(iii) a central simple associative algebra of dimension 9 over \( F \),
(iv) \( Fe_1 \oplus B \) where \( B \) is an algebra (with unity element \( 1-e_1 \)) on which a nondegenerate quadratic form permitting composition is defined; that is, one of:

(iv, a) \( Fe_1 \oplus Fe_2 \) (with \( f_1 = 1, f_2 = 2 \) in (1)),
(iv, b) \( Fe_1 \oplus Fe_2 \oplus Fe_3 \),
(iv, c) \( Fe_1 \oplus Z, Z \) a quadratic field over \( F \),
(iv, d) \( Fe_1 \oplus Q, Q \) a (generalized) quaternion algebra over \( F \),
(iv, e) \( Fe_1 \oplus C, C \) a Cayley algebra over \( F \).

The possible dimensions for \( A \) are 1, 2, 3, 5, and 9.

Sufficiency is immediate, and we are concerned throughout the paper with proving the necessity. Our method (a reduction to trace-admissible algebras) makes essential use in Lemma 2 of the assumed finite dimensionality of \( A \).

1. **Cubic forms.** Let \( V \) be a vector space (of possibly infinite dimension) over a field \( F \) of characteristic \( \neq 2, 3 \). A mapping \( x \rightarrow N(x) \) of \( V \) into \( F \) is called a cubic form on \( V \) in case \( N(ax) = a^3N(x) \) for all \( a \in F \) and \( x \in V \), and

\[
(x, y, z) = \frac{1}{6} [N(x + y + z) - N(x + y) - N(x + z) - N(y + z) + N(x) + N(y) + N(z)]
\]

\(^2\) For the definition, see footnote 4.

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is trilinear. Then \( N(x) = (x, x, x) \). We shall say that a symmetric trilinear form \((x, y, z)\) and its associated cubic form are \textit{nondegenerate} in case \((x, y, z) = 0\) for all \(y, z\) in \(V\) implies \(x = 0\).\(^4\)

Assume that a cubic form \(N(x)\) is defined on a nonassociative algebra \(A\) over \(F\), and that \(N(x)\) permits composition: \((xy, xy, xy) = (x, x, x)(y, y, y)\). We linearize this in \(x\) to
\[
(5) \quad (x_1y, x_2y, x_3y) = (x_1, x_2, x_3)N(y),
\]
and linearize (5) in \(y\) to the fundamental relationship
\[
(6) \quad (x_1y_1, x_2y_2, x_3y_3) + (x_1y_2, x_2y_3, x_3y_2) + (x_1y_3, x_2y_1, x_3y_1) = 6(x_1, x_2, x_3)(y_1, y_2, y_3).
\]
Clearly (6) is equivalent to (3) when the characteristic is \(\neq 2, 3\). Also (6) implies
\[
(7) \quad (xy_1, xy_2, xy_3) = N(x)(y_1, y_2, y_3).
\]

Assume that \(A\) contains 1. Define a linear form \(x \mapsto T(x)\) on \(A\) by \(T(x) = 3(x, 1, 1)\) and a quadratic form \(x \mapsto Q(x) = 3(x, x, 1)\). We derive a number of consequences of (6) for future use. Put \(x_1 = x, x_2 = y, x_3 = z, y_1 = a, y_2 = y_3 = 1\) in (6) to obtain
\[
(8) \quad (xa, y, z) + (x, ya, z) + (x, y, za) = (x, y, z)T(a).
\]
That is, a right multiplication \(Ra\) on \(A\) leaves \((x, y, z)\) invariant if \(T(a) = 0\). Symmetrically,
\[
(9) \quad (ax, y, z) + (x, ay, z) + (x, y, az) = T(a)(x, y, z).
\]
Then (8) and (9) imply
\[
(10) \quad ([x, a], y, z) + (x, [y, a], z) + (x, y,[z, a]) = 0
\]
so that \(Ra - L_a\) leaves \((x, y, z)\) invariant for every \(a\) in \(A\).

Replace \(x\) by \(x^2\) and put \(a = x\) in (8) and (9) to obtain
\[
(11) \quad (x^2x, y, z) + (x^2, xy, z) + (x^2, y, zx) = (x^2, y, z)T(x)
\]
and
\[
(12) \quad (xx^2, y, z) + (x^2, xy, z) + (x^2, y, xz) = T(x)(x^2, y, z).
\]
Replace \(x\) by \(x^2\) and put \(a = x^2\) in (9) to obtain

\(^4\) More generally, if \(F\) has characteristic 0 or \(p > n\), and if \(N(x)\) is a form of degree \(n\) with associated \(n\)-linear form \((x_1, x_2, \ldots, x_n)\) obtained by polarization, we shall say that \((x_1, x_2, \ldots, x_n)\) and \(N(x) = (x, x, \ldots, x)\) are nondegenerate in case \((x_1, x_2, \ldots, x_n) = 0\) for all \(x_2, \ldots, x_n\) in \(V\) implies \(x_1 = 0\).
(13) \((x^2 x^2, y, z) + (x^2, x^2 y, z) + (x^2, y, x^2 z) = T(x^2)(x^2, y, z)\).

Put \(z = 1\) in (9) and rewrite to obtain

(14) \((xy, z, 1) + (y, xz, 1) + (y, z, x) = T(x)(y, z, 1)\).

Now \(z = 1\) in (14) implies

(15) \(T(xy) = T(x)T(y) - 6(x, y, 1)\),

so that, in particular,

(16) \(T(x^2) = [T(x)]^2 - 2Q(x)\).

Also (15) implies

(17) \(T(xy) = T(yx)\) for all \(x, y\) in \(A\).

Moreover, (15) and (14) imply

(18) \(T((xy)z) = T(x(yz))\) for all \(x, y, z\) in \(A\).

For \(T((xy)z) - T(x(yz)) = T(xy)T(z) - 6(xy, z, 1) - T(x)T(yz) + 6(x, yz, 1) = -T(xy, z, 1) + (y, xz, 1) + (y, z, x) + (x, yz, 1)\). Using the relationship which results from interchange of \(x\) and \(z\) in (14), we have

(19) \(T((xy)z) - T(x(yz)) = 6 - (x, y, 1)T(z) + (y, xz, 1) + (y, z, x) + (x, yz, 1)\).

Next put \(x_1 = x^2, x_2 = x_3 = x, y_1 = y, y_2 = z, y_3 = 1\) in (6); using (7), we have

(20) \((x^2 y, xz, x) + (x^2 z, xy, x) = N(x)T(x)(1, y, z) - N(x)(x, y, z)\).

Also, putting \(x_1 = y_1 = x = x, x_2 = y, x_3 = z, y_3 = 1\) in (6) we have

(21) \((x^2, yx, z) + (x^2, y, zx) = Q(x)(x, y, z) - N(x)(1, y, z)\)

by (5). Symmetrically, we have

(22) \((x^2, xy, z) + (x^2, y, xz) = Q(x)(x, y, z) - N(x)(1, y, z)\).

Put \(x_1 = y_1 = x, x_2 = x^2, x_3 = 1, y_2 = y, y_3 = z\) in (6) to obtain

(23) \((x^2, x^2 y, z) + (x^2, x^2, z, y) + (xy, x^2 x, z) + (xy, x^2 z, x)

+ (xz, x^2 x, y) + (xz, x^2 y, x) = 6(x, x^2, 1)(x, y, z)\).

Finally put \(x_1 = x_2 = x, y_1 = y_2 = y, x_3 = y_3 = 1\) in (6) to obtain

(24) \(Q(xy) + 6(xy, x, y) = Q(x)Q(y)\).

Assume henceforth that \(N(x)\) is nondegenerate. Then (3) implies \(N(1) = 1\), and therefore \(T(1) = Q(1) = 3\). Also \(x^3\) is uniquely defined for any \(x\) in \(A\), and
(24) \[ x^3 - T(x)x^2 + Q(x)x - N(x)1 = 0; \]

that is, \( A \) is a cubic algebra. For (11) and (20) imply \( (x^2x - T(x)x^2 + Q(x)x - N(x)1) = 0 \) for all \( y, z \) in \( A \), or \( x^2x - T(x)x^2 + Q(x)x - N(x)1 = 0 \). Similarly (12) and (21) imply \( xx^2 - T(x)x^2 + Q(x)x - N(x)1 = 0 \), so that \( x^2x = xx^2 \) and (24) holds. Also

(25) \[ x^2x^2 = x^2x(= xx^2) \]

for \( x \) in \( A \).

For (13) and (22) imply \( (x^2x^2 - T(x^2)x^2, y, z) = (xy, z, x^3) + (y, xz, x^3) + (xz, xy, x) - 6(x, x^2, 1)(y, z) \). Now \( (xy, z, x^3) + (y, xz, x^3) = (Q(x)x^2 - N(x)x, y, z) \) by (24), (21), (8), and (14). Also \( 6(x, x^2, 1) = Q(x)T(x) - 3N(x) \) by (15) and (24). Hence (16) implies

(26) \[ x^2x^2 = \left\{ [T(x)]^2 - Q(x) \right\} x^2 + [N(x) - Q(x)T(x)]x + N(x)T(x)1. \]

Then (24) and (26) imply (25). Hence \( A \) is power-associative by

**Lemma 1.** Any cubic algebra in which (25) is satisfied is power-associative.

**Proof.** Let \( x^2x = xx^2 = ax^2 + \beta x + \gamma 1 \), and define \( x^i = x^{i-1}x = \alpha_i x^2 + \beta_i x + \gamma_i 1 \). It is well-known that any quadratic algebra is power-associative, so we may assume that \( x^2, x, 1 \) are linearly independent. Then \( \alpha_0 = 0, \beta_1 = 1, \gamma_1 = 0, \alpha_i = \alpha_{i-1} + \beta_{i-1}, \beta_i = \alpha_{i-1} + \gamma_{i-1}, \gamma_i = \alpha_{i-1} + \gamma \), and it is readily established by induction on \( j \) and use of (25) that \( x^i x^j = x^{i+j} \).

2. **Trace-admissibility.** Under the assumptions of §1 we have seen that \( A \) is a power-associative algebra with \( 1 \) on which a linear form \( T(x) \) is defined satisfying (17) and (18). Then the bilinear form \( B(x, y) = T(xy) \) is an admissible trace function [4] for \( A: B(x, y) = B(y, x), B(xy, z) = B(x, yz), B(e, e) \neq 0 \) for any idempotent \( e \), \( B(x, y) = 0 \) if \( xy \) is nilpotent. For we can prove \( T(e) \neq 0 \) for any idempotent \( e \) and \( T(z) = 0 \) for any nilpotent element \( z \).

Assume \( e \neq 1 \) since \( T(1) = 3 \neq 0 \). Then \( e \) and \( 1 \) are obviously linearly independent and

(27) \[ Q(e) - T(e) + 1 = 0, \quad N(e) = 0 \quad \text{if } e \neq 1, \]

by (24). Then (16) and (27) imply

(28) \[ \text{either } T(e) = 1 \text{ or } T(e) = 2 \quad \text{if } e \neq 1. \]

Now \( z = 0 \) implies \( T(z) = N(z) = 0 \). Then \( z^r = 0, r > 1 \), implies \( N(z) = 0 \) by (3). If \( z^2 = 0, \) but \( z \neq 0, \) then (24) implies \( Q(z) = 0 \) so that \( T(z) = 0 \) by (16). We may assume therefore that \( z^2 \neq 0 \), and let \( r \) be the least exponent such that \( z^r = 0, z^{r-1} \neq 0, r > 2 \). Then \( z^{r-1} \) and \( z^{r-2} \) are
obviously linearly independent, so that (24) implies $T(z) = Q(z) = 0$ (giving $z^3 = 0$).

Also $B(x, y)$ is nondegenerate on $A$. For, if $B(x, y) = 0$ for every $y$ in $A$, then $B(x, 1) = T(x) = 0$. Also $B(x, yz) = 0$ for all $y, z$ in $A$, implying $B(xy, z) = B(x, y)T(z) - 6(xy, z, 1) = -6(xy, z, 1) = 0$ by (15). Interchange $y$ and $z$ to obtain $(xz, y, 1) = 0$, so that $(x, y, z) = 0$ for all $y, z$ in $A$ by (14). Then $x = 0$.

If we now assume that $A$ is of finite dimension over $F$, we can apply [4] to obtain

**Lemma 2.** Let $A$ be a finite-dimensional nonassociative algebra with 1 over $F$ of characteristic $\neq 2, 3$. There is a nondegenerate cubic form $N(x)$ on $A$ permitting composition only if $A$ is a separable algebra $A = A_1 \oplus \cdots \oplus A_r$ in which each simple ideal $A_i$ is one of the following: a (commutative) Jordan algebra, a quasiasociative algebra [3, Chapter V], or a flexible quadratic algebra (with the attached commutative algebra $A_i$ a simple Jordan algebra of degree 2).

We shall first sharpen this result considerably in case $r > 1$. Write $x = x_1 + \cdots + x_r, x_i$ in $A_i$, and $1 = e_1 + \cdots + e_r$ where $e_i$ is the unity element of $A_i$ ($e_i \neq 1$ in case $r > 1$). Consider $N_i(x_i) = N(x_i - e_i + 1)$. Then $N(x) = N_1(x_1) \cdots N_r(x_r)$ and

$$N_i(x_i y_i) = N_i(x_i)N_i(y_i).$$

For

$$N_1(x_1) \cdots N_r(x_r) = N(x_1 - e_1 + 1) \cdots N(x_r - e_r + 1)$$

$$= N((x_1 - e_1 + 1) \cdots (x_r - e_r + 1))$$

(associative product!)

$$= N(x_1 + \cdots + x_r - (e_1 + \cdots + e_r) + 1) = N(x_1 + \cdots + x_r),$$

while

$$N_i(x_i y_i) = N(x_i y_i - e_i + 1) = N((x_i - e_i + 1)(y_i - e_i + 1)) = N_i(x_i)N_i(y_i).$$

Now

$$N_i(x_i) = (x_i - e_i + 1, x_i - e_i + 1, x_i - e_i + 1)$$

$$= N(x_i) - 3(x_i, x_i, e_i) + Q(x_i) + 3(x_i, e_i, e_i) + Q(e_i)$$

$$+ T(x_i) - T(e_i) - N(e_i) + N(1) - 6(x_i, e_i, 1).$$

If $e_i \neq 1$, then $N(x_i) = N(x_i e_i) = N(x_i)N(e_i) = 0$ and $Q(e_i) - T(e_i) + N(1) = 0$ by (27). Put $a = e_i$, $x = y = x_i$, $z = 1$ in (8) and $x = x_i$, $y = e_i$ in (23) to obtain $2Q(x_i) + 3(x_i, e_i, e_i) = Q(x_i)T(e_i)$ and $Q(x_i) + 6(x_i, e_i, e_i) = Q(x_i)Q(e_i)$. Then (27) implies

$$2Q(x_i) = Q(x_i)T(e_i), \quad (x_i, e_i, e_i) = 0 \quad \text{if} \quad e_i \neq 1.$$
Put $a = x_i$, $x = y = e_i$, $z = 1$ in (8) and $x = x_i$, $y = e_i$ in (15) to obtain
\[6(x_i, e_i, 1) + 3(e_i, e_i, x) = Q(e_i)T(x_i)\] and $T(x_i) = T(x_i)T(e_i) - 6(x_i, e_i, 1)$. Then (27) implies
\[(31) \quad 6(x_i, e_i, 1) = Q(e_i)T(x_i), \quad (e_i, e_i, x) = 0 \quad \text{if} \quad e_i \neq 1.\]

Hence $N_i(x_i) = Q(x_i) + T(x_i) - Q(e_i)T(x_i)$ if $e_i \neq 1$. By (28) there are two cases. If $T(e_i) = 1$, then $Q(x_i) = 0$ by (30) and $N_i(x_i) = T(x_i)$. If $T(e_i) = 2$, then $Q(e_i) = 1$ by (27) and $N_i(x_i) = T(x_i)$. Thus, if $r > 1$, we can order the simple $A_i$ so that $T(e_i) = 1$ for $i = 1, \cdots, t$, $T(e_i) = 2$ for $i = t + 1, \cdots, r$ (0 $\leq t \leq r$). Then $N(x) = N_i(x_i) \cdots N_r(x_r)$ $= T(x_1) \cdots T(x_1)Q(x_{i+1}) \cdots Q(x_r)$ is of degree $t + 2(r - t) = 2r - t = 3$.

The only possibilities are $r = 2$, $t = 1$, and $r = t = 3$. That is, if $A$ is not simple, then either $A = A_1 \oplus A_2$ with $T(e_i) = 1$, $N_i(x_i) = T(x_i)$, $T(e_2) = 2$, $Q(e_2) = 1$, $N_2(x_2) = Q(x_2)$, or $A = A_1 \oplus A_2 \oplus A_3$ with $T(e_i) = 1$, $N_i(x_i) = T(x_i)$ for $i = 1, 2, 3$.

Consider the situation where $N_i(x_i) = T(x_i)$. By (29) we have $B(x_i, y) = T(x_iy) = T(x_i)T(y_i) = T(x_i)T(e_iy) = T(x_i)B(e_i, y)$ $= B(T(x_i)e_i, y)$ for every $y \in A$. Hence $x_i = T(x_i)e_i$ for every $x_i$ in $A_i$, or $A_i = Fe_i$. Thus, if $A$ is not simple, we have case (iv) of our Theorem as soon as we verify that $N_2(x_2) = Q(x_2)$ is nondegenerate on $A_2$. Suppose that
\[
(x_2, y_2) = \frac{1}{2} \left[ Q(x_2 + y_2) - Q(x_2) - Q(y_2) \right] = 3(x_2, y_2, 1) = 0
\]
for all $y_2 \in A_2$. Then $0 = 2(x_2, e_2) = 6(x_2, e_2, 1) = Q(e_2)T(x_2) = T(x_2)$ by (31), and $B(x_2, y) = T(x_2y) = T(x_2y_2) = T(x_2)T(y_2) - 6(x_2, y_2, 1)$ $= -2(x_2, y_2) = 0$ for all $y \in A$ by (15), implying $x_2 = 0$.

3. Simple algebras. We are left with the case where $A$ is simple. Let $K$ be a splitting field for $A$. Since all of our results are valid for $A_K$, we know that one of the following is true: $A_K$ is simple (implying that $A$ is central simple over $F$), $A_K = K e_1 \oplus S$ (implying, since the simple components of $A_K$ are all isomorphic, that $S = K e_2$ and that $A$ is a quadratic field over $F$, a possibility to be eliminated in Lemma 3), or $A_K = K e_1 \oplus K e_2 \oplus K e_3$ (implying that $A$ is a cubic field over $F$, which is case (ii) of the Theorem).

Suppose that $A$ is central simple, and that $K$ is the algebraic closure of $F$. Since each element of $A_K$ satisfies an equation of degree 3 (or lower) with coefficients in $K$, Lemma 2 implies that $A_K$ is one of: (a) a split central simple (commutative) Jordan algebra of degree 3 (dimension 6, 9, 15, or 27); (b) a split central simple quasiassociative

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6 See footnote 5.
algebra of degree 3; (c) a split central simple flexible quadratic algebra (with the attached commutative algebra $A_K^+$ a split central simple Jordan algebra of degree 2, dimension $\geq 3$); (d) $K_1$ (implying $A = F1$, case (i) of the Theorem).

In (a) and (b) we may represent the elements of $A_K$ by certain $3 \times 3$ matrices; in each case the set $S$ of all $3 \times 3$ symmetric matrices with elements in $K$ is included. Multiplication in $A_K$ is defined by $xy = \lambda x \circ y + (1 - \lambda) y \circ x$ for some $\lambda \in K$ ($\lambda = 1/2$ for the algebras (a)) where $x \circ y$ denotes the ordinary matrix product. Powers of elements of $S$ coincide with ordinary matrix powers. Consider

$$
eq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad 
\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

in $S$. Now $e \neq 1$ is an idempotent, and $x = e(z - e) = ez - e = \lambda e \circ z + (1 - \lambda) z \circ e - e$ is the matrix

$$x = \begin{bmatrix} -1 & \lambda & 0 \\ 1 - \lambda & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$ But $x$ satisfies the equation $x^3 + 2x^2 + (\lambda^2 - \lambda + 1)x + (\lambda^2 - \lambda - 1) = 0$ and, if $\lambda \neq 0, 1$, no equation of lower degree. Hence $N(x) = \lambda - \lambda^2 = N(e(z - e)) = N,eN(z - e) = 0$ by (27), a contradiction unless $\lambda = 0$ or $\lambda = 1$. This eliminates (a) and leaves for (b) only an associative algebra. For the corresponding algebras $A$ we have case (iii) of our Theorem.

Quadratic fields and possibility (c) above are eliminated in

**Lemma 3.** A nondegenerate cubic form $N(x)$ permitting composition cannot be defined on a quadratic field $A$ over $F$ of characteristic $\neq 2, 3$, or on an algebra $A$ over $F$ for which the attached commutative algebra $A^+$ is a central simple Jordan algebra of degree 2.

**Proof.** For $x$ in $A$ satisfies $x^2 = t(x)x - n(x)1$ with $t(x), n(x)$ in $F$, $t(\alpha 1) = 2\alpha$, $n(\alpha 1) = \alpha^2$. Also $A = F1 + M$ where $M$ consists of all $x_0$ satisfying $t(x_0) = 0$. If $A$ is a quadratic field, $M$ contains $u_0$ with $u_0^2 = \gamma 1 = -n(u_0)1$, $\gamma$ a nonsquare in $F$. In the other case, there is a nondegenerate quadratic form $f [5, \S 13]$ on $M$ of dimension $\geq 2$ such that $x_0^2 = f(x_0)1 = -n(x_0)1$ for $x_0 \in M$. Now (24) implies $[T(x) - t(x)]x^2 + [n(x) - Q(x)]x + N(x)1 = 0$. Write $L(x) = T(x) - t(x)$. Then $[n(x) - Q(x) + L(x)t(x)]x + [N(x) - L(x)n(x)]1 = 0$. Whether or not $x \in F1$, we have $Q(x) = n(x) + L(x)t(x)$. Then (16) implies $n(x) = L(x)[t(x) - L(x)]$, so that $n(x_0) = -[L(x_0)]^2$ for all $x_0 \in M$. Thus
\[ \gamma = -\eta(u_0) = [L(u_0)]^2, \] a contradiction, and the nondegenerate quadratic form \( f(x_0) = -\eta(x_0) = [L(x_0)]^2 \) is the square of a linear form \( L(x_0) \) on \( M \), implying that \( M \) is 1-dimensional, a contradiction. This completes the proof of Lemma 3 and of the Theorem.

**Remark.** If \( A \) does not contain 1, it is possible to pass easily (as in [7, p. 957; 6, p. 56]) to an isotopic algebra \( A^* \) with 1. Briefly: \( N(u) \neq 0 \) implies by (5) and (7) that \( x \to xa = xR_a \) and \( x \to ax = xL_a \) are (1-1) for \( a = u^3/N(u) \). By finite-dimensionality we can define multiplication in \( A^* \) by \( x \ast y = (xR_a^{-1})(yL_a^{-1}) \). Then \( a^2 \) is a unity element for \( A^* \) and \( N(x \ast y) = N(x)N(y) \). Thus, without assuming \( 1 \in A \), we have \( \dim A = 1, 2, 3, 5, \) or 9.

**References**


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