ON LIMITS OF MODULE-SYSTEMS

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1. Introduction. This paper is concerned with a generalization of direct and inverse limits. The construction uses local sections defined on sets of a filter $\mathcal{F}$ on $M$. Usually, $M$ is a directed set (see [1]). The purpose of the order-relation is a twofold one: First it serves to generate a filter on $M$, second it is responsible for the existence of certain homomorphisms $\pi^g_\alpha$. That both tasks can be separated was already pointed out in [2]; that only the filtering property of the order relation is necessary for the construction of limits will be shown here.

2. Definition of module-systems.

Definition 1. A pair $(f, g)$, where $f$ is a surjective map of $E$ into $M$ and $g$ a map of $M^2$ into $\mathcal{F}(E^2)$, defines on a pair $(E, M)$ of sets a structure of a module-system over the ring $R$ if it possesses the following properties:

(SM1) For each $\alpha \in M$, the reciprocal image $f^{-1}(\alpha)$ of $\alpha$ under $f$ is a module over $R$;

(SM11) For each pair $(\alpha, \beta) \in M^2$, the part $g(\alpha, \beta)$ of $E^2$ is a sub-module of the product-module $f^{-1}(\alpha) \times f^{-1}(\beta)$ over $R$;

(SM111) For each $\alpha \in M$, the set $g(\alpha, \alpha)$ is a part of the diagonal of $f^{-1}(\alpha) \times f^{-1}(\alpha)$.

Write $E_\alpha$ instead of $f^{-1}(\alpha)$.

Example. Let $E$ be the sum of the sets $E_\alpha$ ($\alpha \in M$) of a direct (resp. inverse) system of $R$-modules over the directed set $M$ and $f$ the map of $E$ into $M$ which assigns to each $x \in E_\alpha$ the element $\alpha \in M$. If $\alpha \leq \beta$, denote with $g(\alpha, \beta)$ (resp. $g(\beta, \alpha)$) the graph of the homomorphism $\pi^g_\alpha$, if not $\alpha \leq \beta$, let $g(\alpha, \beta) = E_\alpha \times E_{\beta}$. The pair $(E, M)$ together with $(f, g)$ forms an $R$-module-system.

Definition 1 can readily be modified in order to suit any algebraic structure, e.g. group-system, vector-space-system.

3. Construction of limit-modules. Let $(E, M)$ be a module-system and $\mathcal{F}$ a filter on $M$. A function $s = (S, A, E)$ defined on a set $A \in \mathcal{F}$ with values in $E$ is a local section of $f$, if the composition $f \circ s$ is the canonical injection of $A$ into $M$. The section $s$ is said to be $g$-admissible if $(\alpha, \beta) \in M^2$ implies $(s(\alpha), s(\beta)) \in g(\alpha, \beta)$.

Denote the set of the $g$-admissible local sections of $f$, defined on a

Presented to the Society, January 21, 1959; received by the editors February 8, 1959.

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set \( A \in \mathfrak{F} \), by \( S(\mathfrak{F}, f, g) \). The relation \( \ll \) for some \( A \in \mathfrak{F} \), \( s_A = t_A \)\( \gg \) is an equivalence relation \( R_\infty \) in \( S(\mathfrak{F}, f, g) \); write \( S_\infty(\mathfrak{F}, f, g) \) for \( S(\mathfrak{F}, f, g)/R_\infty \). We shall endow \( S_\infty(\mathfrak{F}, f, g) \) with the structure of a module over the ring \( R \).

If \( s = (S, A, E) \), \( t = (T, B, E) \) are \( g \)-admissible local sections of \( f \), define \( s + t \) by \( \alpha \to s(\alpha) + t(\alpha) (\alpha \in A \cap B) \), \( \lambda \cdot s \) by \( \alpha \to \lambda \cdot s(\alpha) (\alpha \in A) \), which is possible by axiom (SM_1). Clearly, \( s + t \) and \( \lambda \cdot s \) are local sections which are \( g \)-admissible by axiom (SM_{II}). Note, that addition and multiplication are compatible with \( R_\infty \). Therefore it is allowed to pass to the quotient, thus obtaining a structure \( \mathfrak{M}_\infty \) of a module over the ring \( R \). We call \textit{limit} of the \( R \)-module-system \( (E, M) \) with respect to the filter \( \mathfrak{F} \) and denote by \( \lim_{\mathfrak{F}} (f, g) \) the quotient \( S_\infty(\mathfrak{F}, f, g) \), endowed with the structure \( \mathfrak{M}_\infty \).

**Example.** In the case of a direct system, let \( \mathfrak{F} \) be the filter of the sections of the direct set \( M \neq \emptyset \), in the case of an inverse system, take the reduced set \( \{M\} \). The module \( \lim_{\mathfrak{F}} (f, g) \) is isomorphic to the direct (resp. inverse) limit as was proved by Griffiths (see [2]).


**Definition 2.** Let \( \varphi = (\Phi, E, E') \) and \( \psi = (\Psi, M, M') \) be functions. The pair \( (\varphi, \psi) \) is called a map of the module system \( (E, M) \) into the module-system \( (E', M') \) over the same ring \( R \), if it satisfies the following conditions:

1°. \( \psi \circ f = f' \circ \varphi \);

2°. For each pair \( (\alpha, \beta) \in M^2 \), \( (x, y) \in g(\alpha, \beta) \) implies \( (\varphi(x), \varphi(y)) \in g'(\psi(\alpha), \psi(\beta)) \);

3°. For each \( \alpha \in M \), the map \( \varphi_\alpha \) of \( E_\alpha \) into \( E'_{\psi(\alpha)} \), induced by \( \varphi \), is a homomorphism.

The composition of two maps is a map; for bijections \( \varphi, \psi \) the condition \( \ll (\varphi, \psi) \text{ and } (\varphi^{-1}, \psi^{-1}) \text{ are maps} \gg \) is equivalent to the condition \( \ll (\varphi, \psi) \text{ is an isomorphism} \gg \).

**Proposition 1.** Let \( (\varphi, \psi) \) be a map of a module-system \( (E, M) \) with filter \( \mathfrak{F} \) into a module-system \( (E', M') \) with filter \( \mathfrak{F}' \). If \( \mathfrak{F}' \) is finer than the filter on \( M' \) generated by the image of \( \mathfrak{F} \) under \( \psi \), then the pair \( (\varphi, \psi) \) induces a homomorphism \( (\varphi, \psi)_\infty \) of \( \lim_{\mathfrak{F}} (f, g) \) into \( \lim_{\mathfrak{F}'} (f', g') \).

Let \( s = (S, A, E) \) be a \( g \)-admissible local section of \( f \). Denote by \( s' \) the correspondence \( \varphi \circ s \circ \psi_A^{-1} \), where \( \psi_A \) is the map of \( A \) into \( \psi(A) \) induced by \( \psi \). \( s' \) is a function (Axiom (SM_{III})) which is a \( g' \)-admissible local section of \( f' \). Furthermore, \( s \to s' \) is compatible with the equivalence relations \( R_\infty \) and \( R'_{\infty} \), thus inducing a homomorphism \( (\varphi, \psi)_\infty \) of \( S_\infty(\mathfrak{F}, f, g) \) into \( S_\infty(\mathfrak{F}', f', g') \).
5. **Induced structures.** Let \((E, M)\) be a \(R\)-module-system and \(N\) a part of \(M\). Write \(E_N\) for the reciprocal image of \(N\) under \(f, f_N\) for the subjective map of \(E_N\) into \(N\) induced by \(f\) and \(g_N\) for the map of \(N^2\) into \(\mathfrak{B}(E_N^2)\) induced by \(g\). Obviously, the pair \((f_N, g_N)\) defines an \((E_N, N)\) a structure of a module-system over \(R\), called the induced structure. It is obviously an initial structure in the sense of N. Bourbaki.

**Theorem 1.** Let \((E, M)\) be a module-system, \(N\) a part of \(M\), \(i\) (resp. \(j\)) the canonical injection of \(E_N\) (resp. \(N\)) into \(E\) (resp. \(M\)), and \(\mathcal{F}\) a filter on \(N\). If \(\mathcal{F}'\) is the filter on \(M\) generated by \(\mathcal{F}\), then the map \((i, j)\) induces an isomorphism \((i, j)_{\mathcal{F}}\) of \(\lim_{\mathcal{F}} (f_N, g_N)\) into \(\lim_{\mathcal{F}'} (f, g)\).

By Proposition 1, \((i, j)\) induces a homomorphism which is bijective because of \(s' = i \circ s\) (see proof of Proposition 1).

6. **Cofinality.** A pair \((\alpha, \beta) \in M^2\) is called \(g\)-distinguished if the correspondence \((g(\alpha, \beta), E_\alpha, E_\beta)\) is a function.

**Definition 3.** A filter \(f\)' is cofinal for a filter \(f\) relative to \((f, g)\), if \(f\)' is finer than \(f\), and if for every set \(A \subseteq f\)' there exists a set \(B \subseteq f\) satisfying the following conditions:

1. \((CF_1)\) For every element \(\beta \in B\), there exists an element \(\alpha \in A \cap B\) such that that \((\alpha, \beta)\) is \(g\)-distinguished;
2. \((CF_2)\) If \((\alpha_1, \beta_1)\) and \((\alpha_2, \beta_2)\) are \(g\)-distinguished elements of \((A \cap B) \times B\), then there exists an element \(\alpha \in A \cap B\) such that \((\alpha, \alpha_1)\) is \(g\)-distinguished and the relation

\[ g(\alpha_2, \beta_2) \circ g(\alpha, \alpha_2) \subset g(\beta_1, \beta_2) \circ g(\alpha_1, \beta_1) \circ g(\alpha, \alpha_1) \]

holds (see diagram below).

A filterbase \(\mathcal{B}'\) is cofinal for \(\mathcal{F}\), if the filter \(\mathcal{F}'\) generated by \(\mathcal{B}'\), is cofinal for \(\mathcal{F}\).

**Theorem 2.** Let \((E, M)\) be a module-system, \(i\) (resp. \(j\)) the identity of \(E\) (resp. \(M\)) and \(\mathcal{F}, \mathcal{F}'\) filters on \(M\). If \(\mathcal{F}'\) is cofinal for \(\mathcal{F}\), then the map \((i, j)\) induces an isomorphism \((i, j)_{\mathcal{F}}\) of \(\lim_{\mathcal{F}} (f, g)\) into \(\lim_{\mathcal{F}'} (f, g)\).

By Proposition 1, \((i, j)\) induces a homomorphism which is bijective because of \(s' = s\) (see proof of Proposition 1).
Theorem 3. Let \((f, g), (f', g')\) be pairs of maps, defining structures of \(R\)-module-systems on \((E, M), \mathcal{F}, \mathcal{F}'\) filters on \(M\) and \(N\) a part of \(M\). Suppose, that the induced structures and filters on \(N\) exist and coincide, and that \(\mathcal{F}_N\) (resp. \(\mathcal{F}'_N\)) is cofinal for \(\mathcal{F}\) (resp. \(\mathcal{F}'\)) relative to \((f, g)\) (resp. \((f', g')\)). Then the modules \(\operatorname{lim}^\mathcal{F}_\mathcal{A} (f, g)\) and \(\operatorname{lim}^\mathcal{F}'_\mathcal{A} (f', g')\) are isomorphic.

Write \(\mathcal{G}\) (resp. \(\mathcal{G}'\)) for the filter on \(M\) generated by \(\mathcal{F}_N\) (resp. \(\mathcal{F}'_N\)). Consider the diagram

\[
\operatorname{lim}^\mathcal{G}_\mathcal{A} (f, g) \rightarrow \operatorname{lim}^\mathcal{G}_\mathcal{A} (f, g) \leftarrow \operatorname{lim}^\mathcal{G}_N (f_N, g_N).
\]

The first map is induced by a pair of identities and is an isomorphism according to Theorem 2, the second map is induced by a pair of canonical injections and is an isomorphism according to Theorem 1. Because the same argument holds for the primed maps, the theorem follows.

References
