1. Introduction. Let $E$ be a real Banach space, and $V$ a (proper or improper) subset of $E$. Let $D(x, h)$ be a function defined for $x$ in $V$ which for fixed $x$ is a linear continuous functional on $E$ in the variable $h$, in other words, let $D(x, h)$ define a map of $V$ into the space $E^*$ conjugate to $D$. Let $E$ and $E^*$ have the topologies induced by their respective norms. The map $D(x, h)$ is called completely continuous if it is continuous and if the image of each bounded subset of $V$ has a compact closure in $E^*$. It is said to be a gradient mapping if there exists a scalar function $I(x)$ such that $D(x, h)$ is the Fréchet differential of $I(x)$ at the point $x$. For the motivation of this terminology and for details, we refer the reader to [6] or [7]. It is known that there is a close connection between the complete continuity of the gradient on the one hand and the weak continuity or related properties of the scalar $I(x)$ on the other hand [6, Theorems 3.2 and 3.3]. Further literature is quoted in [7, footnote 11]; see also [2] and [3]. In particular, it is known [6, Theorem 3.3] that for convex $V$ the complete continuity of $D(x, h)$ implies the following property of the scalar $I(x)$: to each positive $\eta$ there corresponds a finite number of elements $l_i (i=1, 2, \cdots, N)$ of $E^*$ such that the inequalities

\begin{equation}
|l_i(h)| < \eta\|h\|/2 \quad (i = 1, 2, \cdots, N)
\end{equation}

imply

\begin{equation}
|I(x + h) - I(x)| < \eta\|h\|, \quad h, x + h \text{ in } V.
\end{equation}

For the case where $E$ is a Hilbert space, the converse of this theorem was stated and proved in [6]. The reason for the restriction was as follows: Hilbert spaces are the only Banach spaces (of dimension at least 3) with the property that there exists a projection of norm 1 on every closed linear subspace (see [5]), and such projections were used in the proof given in [6].

However, it will be seen in the present note, that for the purpose at hand it is not necessary to have projections of norm 1 (or of uniformly bounded norm) on all closed linear subspaces; rather, it will be sufficient that such projections exist on a large enough collection of subspaces. To be more precise we introduce the following concept.

Received by the editors October 7, 1958 and, in revised form, February 6, 1959.

1 The paper was written while the author was recipient of a John Simon Guggenheim Memorial Fellowship.

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Definition 1.1. Let $E$ be a Banach space and $E^*$ the space conjugate to $E$. We say that $E$ has property $P$ if there exists a set $\{f_i\}$ $(i = 1, 2, \cdots)$ of linearly independent elements of $E^*$ and a positive number $M$ with the following two properties: the finite linear combinations of elements of $\{f_i\}$ are dense in $E^*$ (that is, $\{f_i\}$ is a fundamental set in $E^*$ in the sense of Banach [1, p. 58]), and if $N_i = \{x \in E | f_i(x) = 0\}$, then for each integer $n$ there exists a projection of norm at most $M$ on the intersection $\cap_n N_i$.

In §2 we shall show that the converse theorem in question is true under the assumption $P$; that is we shall prove the following

Theorem 1.1. Let $E$ be a Banach space with property $P$. Let $I(x)$ be a scalar defined in a convex subset $V$ of $E$ with the following two properties: $I(x)$ has a Fréchet differential which is continuous in $x$, and corresponding to each positive $\eta$ there exists a finite number of elements $l_1, l_2, \cdots, l_n$ of $E^*$ such that

\[(1.3) \quad |I(x + h) - I(x)| < \eta\|h\| \quad (h, x + h \text{ in } V)\]

for all $h$ for which

\[(1.4) \quad |l_i(h)| \leq \eta\|h\| \quad (i = 1, 2, \cdots, n).\]

Then $D(x, h)$ is completely continuous.

§3 deals with conditions that are sufficient for a Banach space $E$ to have property $P$. In particular, every reflexive Banach space with a base will be seen to have property $P$.

2. Proof of Theorem 1.1. We first recall some properties of an arbitrary Banach space $E$. Let $l_1, l_2, \cdots, l_p$ be $p$ linearly independent elements of $E^*$, and let $N = \{x \in E | l_i(x) = 0, i = 1, 2, \cdots, p\}$. Moreover let $a_1, a_2, \cdots, a_p$ be any set of $p$ elements of $E$ which are linearly independent mod $N$. (This means that $\sum_{i=1}^{p} a_i \gamma_i \in N$ implies that the real numbers $\gamma_i$ are all $0$.) Finally let $E^p$ be the space spanned by the $a_i$. Then each $x$ in $E$ admits a unique decomposition

\[(2.1) \quad x = x_1 + n \quad (x_1 \in E^p, n \in N),\]

and, on account of the linear independence of the $l_i$, the determinant of the $l_i(a_j)$ is different from zero. It is easily seen that this latter fact implies the existence of a base $b_1, b_2, \cdots, b_p$ of $E^p$ which is "orthogonal," that is, for which

\[(2.2) \quad l_i(b_j) = \delta_{ij} \quad (i, j = 1, 2, \cdots, p),\]

where $\delta_{ij}$ is the Kronecker symbol. If then $l(x)$ is an arbitrary linear
continuous functional on \( E \), one obtains almost immediately the representation

\[
(2.3) \quad l(x) = \sum_{j=1}^{p} l_j(x) l(b_j) + l(n),
\]

(apply \( l \) to (2.1) after \( x_i \) has been expressed in terms of the \( b_j \), and use (2.2)). With the notation of this paragraph, the following lemma is an immediate consequence of the representation (2.3); it may be considered as a generalization of the well-known fact that any \( l \) vanishing on \( N \) is a linear combination of the \( l_i \).

**Lemma 2.1.** Let \( l(x) \) have the property that there exists a positive number \( \eta \) such that

\[
(2.4) \quad |l(y)| < \eta \|y\| \quad \text{for all } y \text{ in } N.
\]

Then, for all \( x \) in \( E \),

\[
(2.5) \quad l(x) = \sum_{i=1}^{p} \alpha_i l_i(x) + R(x)
\]

with suitable constants \( \alpha_i \) and with

\[
(2.6) \quad |R(x)| \leq M \eta \|n\|,
\]

where \( n \) is the element of \( N \) appearing in the decomposition (2.1) of \( x \).

In addition to Lemma 2.1 we shall need the following

**Lemma 2.2.** If the assumption in Lemma 2.1 holds, and if there exists a projection \( \pi \) of \( E \) on \( N \) of norm at most \( M \), then the representation (2.5) holds with

\[
(2.7) \quad |R(x)| \leq M \eta \|x\|.
\]

**Proof.** Since the elements \( b_i \) are linearly independent mod \( N \), the elements \( b'_i = b_i - \pi b_i \) have the same property. If then \( \tilde{E}^p \) is the space spanned by the \( b'_i \), we obtain instead of (2.1) the decomposition

\[
(2.8) \quad x = x'_i + n' \quad (x'_i \in \tilde{E}^p, n' \in N).
\]

It is easily verified that \( x'_i = x - \pi(x) \) and \( n' = \pi(x) \), and consequently, by assumption,

\[
(2.9) \quad \|n'\| \leq \tilde{M} \|x\|.
\]

If we now apply Lemma 2.1 to the new decomposition (2.8), we have to replace \( n \) by \( n' \) in (2.6). The inequality thus obtained together with (2.9) proves (2.7).
We now turn to the proof of Theorem 1.1. We have to establish the complete continuity of the gradient mapping $D(x, h)$. For this purpose we may without loss of generality assume that $V$ is bounded. Therefore, by [6, Lemma 3.2] it will be sufficient to prove that, corresponding to a given $\epsilon > 0$, there exists a mapping $D'(x, h)$ of $V$ into $E^*$ with the following two properties: the image of $V$ under $D'$ is contained in a finite-dimensional subspace of $E^*$, and

$$\left| D(x, h) - D'(x, h) \right| \leq \epsilon \|h\|.$$  

We first choose, corresponding to the given $\epsilon$,

$$\eta = \epsilon / M,$$

where $M$ is the number appearing in Definition 1.1. Corresponding to $\eta$, there exist by assumption elements $l_i$ of $E^*$ such that (1.4) implies (1.3). Now, by Definition 1.1, the set \{f_i\} is fundamental in $E^*$. Consequently there exist finite linear combinations $f'_i$ of elements of the set \{f_i\} such that $\|l_i - f'_i\| < \eta$, in other words, such that

$$\left| l_i(h) - f'_i(h) \right| < \eta \|h\| \quad (i = 1, 2, \cdots, n).$$

Now let the integer $p$ be such that the set $f_1, f_2, \cdots, f_p$ contains all elements of \{f_i\} which occur in the linear combinations $f'_i \ (i = 1, 2, \cdots, n)$, and let $N = \{h | f_i(h) = 0, i = 1, 2, \cdots, p\}$. Then, for $h$ in $N$, all $f'_i(h)$ are zero, and (2.12) shows that (1.4) holds for such $h$; by assumption this implies (1.3). But from (1.3) together with the definition of the Fréchet differential, one concludes easily (see [6, p. 430]) that

$$\left| D(x, h) \right| \leq \eta \|h\|$$

for all $h$ in $N$. We now may apply Lemma 2.2, since by Definition 1.1 there exists a projection of $E$ on $N$ of norm at most $M$. In the present notation, we thus obtain from (2.5), (2.7), and (2.11)

$$D(x, h) = \sum_{i=1}^{p} \alpha_i f_i(h) + R(h), \quad \left| R(h) \right| \leq M \eta \|h\| \leq \epsilon \|h\|.$$ 

This shows that, with the definition

$$D'(x, h) = \sum_{i=1}^{p} \alpha_i f_i(h),$$

$D'(x, h)$ satisfies the two requirements formulated at the beginning of this proof.
3. **Sufficient conditions for property** $P$. Let $E$ be a Banach space with a base $\{b_i\}$. Then there exists a unique sequence $\{f_i\}$ of elements of $E^*$ such that, for every element $x$ of $E$,

\[(3.1) \quad x = \sum_{i=1}^{\infty} bf_i(x)\]

[1, Chapter VII, §3]. We now make the assumption that the set $\{f_i\}$ is fundamental in $E^*$, and we claim that then $E$ has property $P$. Indeed, if $N^p = \{x \in E | f_i(x) = 0 \text{ for } i = 1, 2, \ldots, p\}$, then $N^p$ is the space spanned by $b_{p+1}, b_{p+2}, \ldots$. It follows from Banach’s results [1, p. 111] that the map $E \to N^p$ mapping the point (3.1) of $E$ into the point

\[\sum_{i=p+1}^{\infty} b_i f_i(x)\]

is a projection with a norm admitting a bound independent of $p$. This shows that $E$ has property $P$.

It follows in particular that every reflexive Banach space with a base has property $P$. For in such spaces, our assumption that the set $\{f_i\}$ is fundamental is satisfied; this follows from [4, Lemma 1, p. 70 together with Theorem 3, p. 71].

For reflexive Banach spaces with a base, Citlanadze [2] stated without proof some propositions related to Theorem 1.1. In a later paper [3, Theorem 1] he proved such a theorem in the more special case of an $L_p$ space ($p > 1$).

**Bibliography**